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On a Class of Cubic Surfaces with Curves of the Same Species.

By John Eiesland.

I.

"In Raume (x, y, z)," says Lie,[*] "kennen wir noch einige Flächen, die unendlich viele Scharen von Curven gleicher Gattung enthalten, allerdings nicht so viele wie die oben besprochene Flächenart." By "die oben besprochene Flächenart" he refers to the tetrahedral symmetrical surfaces

$$x^a y^a z^a = \rho$$

which contain ∞^2 families of curves of the same species.[†] He then mentions two kinds of surfaces having the same property, viz.: the planes

$$Ax + By + Cz + D = 0$$

and the ∞^5 quadric surfaces

$$Ayz + Bxz + Cxy + Lx + My + Nz = 0,$$

the latter having 4 families, namely the rectilinear generators and two families of twisted cubics. In the following I have used an indirect method of obtaining other surfaces of the same kind, no direct method being obvious. The method consists in first finding translation-surfaces of a certain form that correspond to a unicursal quartic, irreducible or not, in the plane at infinity, and then transforming these to the so-called logarithmic space by means of the transformation

$$X = \log x, \qquad Y = \log y, \qquad Z = \log z, \tag{1}$$

the space (X, Y, Z) being the space of the translation-surface. This transformation has been used by Lie for this and other purposes.[‡]

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[*] Lie–Scheffers, "Geometrie der Berührungstransformationen," p. 363.

[†] For definition of the term "species" in the sense here used, see *ibid.*, pp. 333, 334.

[‡] Lie–Scheffers, "Berührungstransformationen," p. 356.

Lie has shown that to all reducible quartic curves in the plane at infinity consisting of two intersecting conics there correspond translation-surfaces of the form,

$$Ae^{Y+Z} + Be^{Z+X} + Ce^{X+Y} + Le^X + Me^Y + Ne^Z = 0, \qquad (2)$$

and all their transforms by a linear projective transformation. The points P and P', where the two tangents at the points of intersection of the two conjugate translation-curves pierce the plane at infinity, must be taken on different conics if the surface shall have a fourfold mode of generation. If P and P' are taken on the same conic we get surfaces that can be generated in ∞ number of ways.

By means of the logarithmic transformation (1) the surface (2) is transformed into a quadric

$$Axy + Bzx + Cxy + Lx + My + Nz = 0, \qquad (3)$$

which has four families of *curves of the same species*,* corresponding to the four families of translation-curves. (See Lie–Scheffers, "Berührungstransformationen," p. 347)., These curves consist of two sets of rectilinear generators and two families of twisted cubics which pass through the vertices of the tetrahedron of reference.

Since the surface (3) contains no edge of the tetrahedron, there exists an involutory transformation

$$x_1 = \frac{\lambda}{x}, \qquad y_1 = \frac{\mu}{y}, \qquad z_1 = \frac{\nu}{z} \qquad (4)$$

for which the surface remains invariant; the two families of cubic curves are transformed into the two sets of twisted cubics and *vice versa*. For proof of this and other theorems we refer the reader to the Lie–Scheffers volume on contact transformations.

In a paper published in Vol. XXX of AM. JOUR. OF MATH.† I have extended the theory of translation-surfaces to the more general case in which the curve in the plane at infinity is an irreducible unicursal quartic. It was found that to such a curve, having real double points with distinct tangents, there corresponds a translation-surface of the form

$$A + Be^X + Ce^Y + De^Z + Ee^{X+Z} + Fe^{X+Y} + Ge^{Z+Y} + He^{X+Y+Z} = 0 \qquad (5)$$

with the following fundamental relation between the coefficients:

$$EGAF = HDCB, \qquad (6)$$

---

*To a translation in the space (X, Y, Z) there corresponds a transformation of the form $x_1 = \lambda x$, $y_1 = \mu y$, $z_1 = \nu z$; hence, to the translation-curves belonging to any one family there corresponds a family of curves of the same species, using Lie's definition in "Theorie der Berührungstransformationen," p. 330.

† "On Translation-Surfaces Connected with a Unicursal Quartic," Vol. XXX, pp. 170–208.

which expresses the property that all surfaces (5) have a *center of symmetry.* Moreover, since there are ∞^3 unicursal quartics (projectively non-equivalent), there will be ∞^3 *types* of translation-surfaces (5) that have a fourfold mode of generation. We shall repeat here a few of the formulas obtained in the above-mentioned paper.

If the origin be taken as center of symmetry, the surface takes the form

$$A(1 - e^{X+Y+Z}) + B'(e^{Y+Z} - e^X) + C'(e^{X+Z} - e^Y) + D'(e^{X+Y} - e^Z) = 0. \quad (5')$$

The parametric representation of (5) is

$$X = \log \frac{(\rho_1 - \alpha_1)(\rho_2 - \alpha_1)}{(\rho_1 - \beta_1)(\rho_2 - \beta_1)}, \quad Y = \log \frac{(\rho_1 - \alpha_2)(\rho_2 - \alpha_2)}{(\rho_1 - \beta_2)(\rho_2 - \beta_2)}, \quad (5'')$$

$$Z = \log (\rho_1 - k)(\rho_2 - k),$$

where α_1, β_1; α_2, β_2 are the roots of the equations

$$\rho_1^2 + 2b\rho + 1 = 0,$$
$$\rho_1^2 + (4b + 2mc)\rho_1 + m^2 + 4b^2 + 4bmc = 0$$

respectively, and $k = \dfrac{-2(b + mc)}{1 - m^2}$, m being any one root of the equation

$$m^2 + 2am + 1 = 0 ; $$

a, b, c are the parameters of the quartic curve

$$x^2 + y^2 - 2axy + x^2y^2 - 2bx^2y - 2cy^2 = 0. \quad (7)$$

The coefficients A, B, ..., H have the following values (AM. JOUR., Vol. XXX, p. 175):

$$A = (\alpha_1 - \alpha_2)(k - \alpha_1)(k - \alpha_2), \quad B = (\alpha_2 - \beta_1)(k - \alpha_2)(k - \beta_1),$$
$$C = (\beta_2 - \alpha_1)(k - \alpha_1)(k - \beta_2), \quad D = \alpha_2 - \alpha_1, \quad E = \beta_1 - \alpha_2, \quad (8)$$
$$F = (\beta_1 - \beta_2)(k - \beta_1)(k - \beta_2), \quad G = \alpha_1 - \beta_2, \quad H = \beta_2 - \beta_1.$$

If now we apply the transformation (1) to the surface (5) we obtain a cubic surface,

$$A + Bx + Cy + Dz + Exz + Fxy + Gyz + Hxyz = 0, \quad (9)$$

which is analogous to the surface (3) obtained by Lie. We propose to study these surfaces and their characteristic properties.

THEOREM I. *Given any surface* (9) *with the identical relation* $AEGF = BCDH$ *between the coefficients, there exists an involutory transformation* (4) *which will leave the surface invariant.*

Proof. Transforming we have

$$Axyz + B\lambda yz + C\mu xz + D\nu yx + E\lambda\nu y + F\lambda\mu z + G\nu\mu x + H\lambda\mu\nu = 0,$$

which will be identical with (9), if we put

$$\lambda = \frac{AG}{BH}, \quad \mu = \frac{EA}{CH}, \quad \nu = \frac{FA}{DH}, \tag{10}$$

taking also account of the relation (6).   Q. E. D.

The point $(\sqrt{\lambda}, \sqrt{\mu}, \sqrt{\nu})$ we shall call the *center of involution*; in fact, to an involution in space x, y, z there corresponds in (X, Y, Z) an inversion (Spieglung) and, since the center of symmetry is midway between the points X, Y, Z and $-X, -Y, -Z$, in the space x, y, z, the corresponding center of involution must be $(\sqrt{\lambda}, \sqrt{\mu}, \sqrt{\nu})$.

Consider the transformation

$$x = \lambda' x_1, \quad y = \mu' y_1, \quad z = \nu' z_1. \tag{11}$$

All surfaces that are transforms of (9) by this transformation we shall call equivalent. In particular, it is possible to find an equivalent surface whose center of involution is $(-1, -1, -1)$, in which case it is of the form

$$A(1 + xyz) + B'(yz + x) + C'(xz + y) + D'(xy + z) = 0; \tag{12}$$

in fact, if we perform the transformation, we have

$$A + B\lambda' x_1 + C\mu' y_1 + D\nu' z_1 + E\lambda'\nu' x_1 z_1 + F\lambda'\mu' x_1 y_1 + G\mu'\nu' y_1 z_1$$
$$+ H\lambda'\mu'\nu' x_1 y_1 z_1 = 0, \tag{13}$$

and putting $A = H\lambda'\mu'\nu'$, $B\lambda' = G\nu'\mu'$, $C\mu' = E\lambda'\nu'$, $D\nu' = F\lambda'\mu'$, we have, using the identity (6),

$$\lambda' = \sqrt{\frac{AG}{BH}}, \quad \mu' = \sqrt{\frac{EA}{CH}}, \quad \nu' = \sqrt{\frac{FA}{DH}}. \tag{14}$$

Substituting these values in (13) we have

$$A(1 + xyz) + B'(yz + x) + C'(xz + y) + D'(xy + z) = 0, \tag{12}$$

where

$$B' = B\sqrt{\frac{AG}{BH}}, \quad C' = C\sqrt{\frac{EA}{CH}}, \quad D' = D\sqrt{\frac{FA}{DH}}.$$

This equation might also have been obtained from (9) by putting

$$x = \frac{\sqrt{\lambda}}{x'}, \quad y = \frac{\sqrt{\mu}}{y'}, \quad z = \frac{\sqrt{\nu}}{z'}.$$

Hence, to the involutory center $(-1, -1, -1)$ of (12) corresponds the center $(\sqrt{\lambda}, \sqrt{\mu}, \sqrt{\nu})$ of (9) as found above.

If we transform (9) to the form

$$A(1 - xyz) + B'(yz - x) + C'(xz - y) + D'(xy - z) = 0,$$

which has the center of involution $(1, 1, 1)$, the involutory transformation is

$$x = \frac{-\sqrt{\lambda}}{x'}, \quad y = \frac{-\sqrt{\mu}}{y'}, \quad z = -\frac{\sqrt{\nu}}{z'}.$$

We have thus proved:

THEOREM II. *It is always possible by means of a transformation* (11) *to transform the surface* (9) *into an equivalent surface of the form*

$$A(1 + xyz) + B(yz + x) + C(xz + y) + D(xy + z) = 0.$$

Theorems I and II are true for any surface of the form (9) provided the identical relation $EGAF = HDCB$ holds. On account of some unimportant but special assumptions made in deriving the surface (5) it appears from the equations (8) that two other identical relations exist. One of these is $D + E + G + H = 0$, the second, also homogeneous, is easily found but may be omitted here. Both are accidental and have no special geometric significance. The relation $EGAF = BCDH$, on the other hand, is fundamental, as was observed before, p. 2. If we assume, à *priori*, the coefficients A, B, ..., H arbitrary and real, we can not say that the parametric representation of (5) is expressed by the equations (5''). It will be shown later how to obtain the parametric representation of (9) when only (6) is true.

Let us consider the surface (9), whose coefficients have the values given by (8). The parametric representation is obtained from (5'') as follows:

$$x = \frac{(\rho_1 - \alpha_1)(\rho_2 - \alpha_1)}{(\rho_1 - \beta_1)(\rho_2 - \beta_1)}, \quad y = \frac{(\rho_1 - \alpha_2)(\rho_2 - \alpha_2)}{(\rho_1 - \beta_2)(\rho_2 - \beta_2)}, \quad (16)$$

$$z = (\rho_1 - k)(\rho_2 - k).$$

If now we use the transformation

$$x = \frac{\lambda}{x_1}, \quad y = \frac{\mu}{y_1}, \quad z = \frac{\nu}{z_1}, \quad (4)$$

where λ, μ, ν have the values

$$\lambda = \frac{AG}{BH}, \quad \mu = \frac{EA}{CH}, \quad r = \frac{FA}{DH}, \quad (10)$$

we obtain the same surface (16) but with a different mode of representation, viz.:

$$x_1 = \frac{\lambda(\rho_3 - \beta_1)(\rho_4 - \beta_1)}{(\rho_3 - \alpha_1)(\rho_4 - \alpha_1)}, \quad y_1 = \frac{\mu(\rho_3 - \beta_2)(\rho_4 - \beta_2)}{(\rho_3 - \alpha_2)(\rho_4 - \alpha_2)},$$

$$z_1 = \frac{\nu}{(\rho_3 - k)(\rho_4 - k)}. \tag{16'}$$

ρ_1, ρ_2, ρ_3 and ρ_4 are the four families of twisted cubics corresponding to the four translation-curves on (5). The involutory transformation (4) has transformed ρ_1 and ρ_2 into ρ_3 and ρ_4, and, if we start with (16'), ρ_3 and ρ_4 into ρ_1 and ρ_2. It may also be observed that the parametric representation of (12) may be obtained by putting $\sqrt{\lambda}, \sqrt{\mu}, \sqrt{\nu}$ for λ, μ, ν in (16') so that (12) may be put in either one of the forms

$$x = \frac{\sqrt{\lambda}(\rho_3 - \beta_1)(\rho_4 - \beta_1)}{(\rho_3 - \alpha_1)(\rho_4 - \alpha_1)}, \quad y = \frac{\sqrt{\mu}(\rho_3 - \beta_2)(\rho_4 - \beta_2)}{(\rho_3 - \alpha_2)(\rho_4 - \alpha_2)},$$

$$z = \frac{\sqrt{\nu}}{(\rho_3 - k)(\rho_4 - k)}; \tag{12'}$$

$$x = \frac{(\rho_1 - \alpha_1)(\rho_2 - \alpha_1)}{\sqrt{\lambda}(\rho_1 - \beta_1)(\rho_2 - \beta_1)}, \quad y = \frac{(\rho_1 - \alpha_2)(\rho_2 - \alpha_2)}{\sqrt{\mu}(\rho_1 - \beta_2)(\rho_2 - \beta_2)},$$

$$z = \frac{(\rho_1 - k)(\rho_2 - k)}{\sqrt{\nu}}. \tag{12''}$$

Let there be given a unicursal quartic in the plane at infinity, and let the tangents at the double points be real and distinct. We write the equation, as before,

$$x^2 + y^2 - 2axy + x^2y^2 - 2bx^2y - 2cy^2 = 0. \tag{9}$$

To it there corresponds the translation-surface (5). To the ∞^3 values of the parameters a, b, c correspond ∞^3 *types* of surfaces (5), since the coefficients $A, \ldots, H$ are functions of these three parameters. Using the logarithmic transformation we obtain the cubic surface (9). We may now extend the term *type* also to these transforms of the ∞^8 translation-surfaces and we shall say that they correspond logarithmically to the ∞^3 non-equivalent quartics in the plane at infinity of the (X, Y, Z) space, so that we may say:

To the ∞^3 projectively non-equivalent unicursal quartics with double points having distinct tangents there correspond in the logarithmic space ∞^8 types of cubic surfaces

$$A + Bx + Cy + Dz + Exz + Fxy + Gyz + Hxyz = 0. \tag{9}$$

By employing the ∞^3 transformations

$$x_1 = \lambda' x, \quad y_1 = \mu' y, \quad z_1 = \nu' z, \tag{11}$$

we obtain ∞^3 surfaces corresponding to the ∞^3 translation-surfaces. Therefore, to any given quartic, (a, b, c being fixed parameters) there corresponds in the logarithmic space ∞^3 cubic surfaces. Hence the

THEOREM III. *To all the ∞^3 non-equivalent unicursal quartics in the plane at infinity there correspond in the logarithmic space ∞^6 cubic surfaces* (9). *These arrange themselves in ∞^3 types, ∞^3 surfaces belonging to the same type. The family of ∞^3 surfaces belonging to each type is invariant for the ∞^3 transformations* (11). *For each surface there exists an involutory transformation*

$$x_1 = \frac{\lambda}{x}, \qquad y_1 = \frac{\mu}{y}, \qquad z_1 = \frac{\nu}{z}, \tag{4}$$

for which the surface remains invariant; the surface contains 4 families of twisted cubics which group themselves in pairs such that each pair belongs to the same species and one pair is by the transformation (4) *transformed into the second pair and vice versa.*

II.

We shall now resume the study of the surfaces (9), assuming that the coefficients A, B, ..., H satisfy the single identical relation $EGAF = HDCB$. In the first place we notice that it has three nodes at the three vertices of the tetrahedron of reference that are situated in the plane at infinity. In order to begin with the most general case we shall assume these to be ordinary nodes. The three edges joining the nodes will lie on the surface, each counting as four, and will account for 12 of the 27 straight lines on the surface. The surface is of the sixth class, since it has three nodes (Salmon, "Geometry of Three Dimensions," pp. 488, 489, fourth ed.).

Let there now be given a surface

$$x = \frac{(a_1\rho_1 + b_1)(a_1\rho_2 + b_1)}{(c_1\rho_1 + d_1)(c_1\rho_2 + d_1)}, \qquad y = \frac{(a_2\rho_1 + b_2)(a_2\rho_2 + b_2)}{(c_2\rho_1 + d_2)(c_2\rho_2 + d_2)},$$

$$z = \frac{(a_3\rho_1 + b_3)(a_3\rho_2 + b_3)}{(c_3\rho_1 + d_3)(c_3\rho_2 + d_3)};$$

introducing new parameters ρ_1', ρ_2' defined by the equations

$$\rho_1' = \frac{a_3\rho_1 + b_3}{c_3\rho_1 + d_3}, \qquad \rho_2' = \frac{a_3\rho_2 + b_3}{c_3\rho_2 + d_3},$$

this surface may be put into the form

$$x = \frac{(\rho_1 - \beta_1)(\rho_2 - \beta_1)}{(\gamma_1\rho_1 - \delta_1)(\gamma_1\rho_2 - \delta_1)}, \qquad y = \frac{(\rho_1 - \beta_2)(\rho_2 - \beta_2)}{(\gamma_2\rho_1 - \delta_2)(\gamma_2\rho_2 - \delta_2)}, \qquad z = \rho_1\rho_2, \tag{17}$$

from which, by eliminating ρ_1 and ρ_2, we get the surface

$$A + Bx + Cy + Dz + Exz + Fxy + Gyz + Hxyz = 0, \qquad (18)$$

where $A, B, \ldots, H$ have the following values:

$$
\begin{aligned}
A &= \beta_1\beta_2(\beta_1 - \beta_2). & B &= \delta_1\beta_2(\beta_2\gamma_1 - \delta_1). & C &= \beta_1\delta_2(\delta_2 - \beta_1\gamma_2), \\
D &= \beta_2 - \beta_1, & E &= \gamma_1(\delta_1 - \beta_2\gamma_1), & F &= \delta_1\delta_2(\delta_1\gamma_2 - \delta_2\gamma_1), \\
G &= \gamma_2(\beta_1\gamma_2 - \delta_2), & H &= \gamma_1\gamma_2(\gamma_1\delta_2 - \delta_1\gamma_2).
\end{aligned}
\qquad (19)
$$

From these equations we easily see that the relation $AEGF = HBCD$ is identically satisfied. *The surface* (17) *therefore belongs to the class of surfaces we are discussing.* The curves ρ_1 and ρ_2 are the twisted cubics of a pair of families belonging to the same species, the second pair, ρ_3 and ρ_4, also of the same species, being obtained by taking the reciprocals of x, y and z and multiplying each by the quantities $\dfrac{AG}{BH}$, $\dfrac{EA}{CH}$, $\dfrac{FA}{DH}$, respectively. (See pp. 5 and 6.)

Conversely, *given a surface* (18) *with real non-vanishing coefficients satisfying the identical relation* (6), *a parametric representation of the form* (17) *may be found.*

In the first place, it is clear, that if in (17) β_1, β_2, γ_1, γ_2, δ_1, δ_2 are all real, the coefficients of (18) will all be real. If, however, (18) be given with real coefficients, it is not to be expected that these quantities will always be real in the parametric representation; but this point will be made clear later. We shall show how to determine (17) when (18) is given. We must have

$$
\begin{aligned}
\rho A &= \beta_1\beta_2(\beta_1 - \beta_2), & \rho B &= \delta_1\beta_2(\beta_2\gamma_1 - \delta_1), & \rho C &= \beta_1\delta_2(\delta_2 - \gamma_2\beta_1), \\
\rho D &= \beta_2 - \beta_1, & \rho E &= \gamma_1(\delta_1 - \beta_2\gamma_1), & \rho F &= \delta_1\delta_2(\delta_1\gamma_2 - \delta_2\gamma_1), \qquad (20) \\
\rho G &= \gamma_2(\beta_1\gamma_2 - \delta_2), & \rho H &= \gamma_1\gamma_2(\gamma_1\delta_2 - \delta_1\gamma_2),
\end{aligned}
$$

where ρ is a factor of proportionality.

This system contains seven independent equations and seven unknowns. We shall show how to solve it. We obtain by division the following simple equations

$$\frac{A}{D} = -\beta_1\beta_2, \quad \frac{B}{E} = -\frac{\delta_1}{\gamma_1}\beta_2, \quad \frac{C}{G} = -\frac{\delta_2}{\gamma_2}\beta_1, \quad \frac{F}{H} = -\frac{\delta_1\delta_2}{\gamma_1\gamma_2}, \qquad (21)$$

of which only three are independent owing to the relation (6). We also find from (20) and (21)

$$\frac{D}{B} = -\frac{\dfrac{A}{D} + \beta_2^2}{\beta_2^2\delta_1^2\left(1 + \dfrac{E}{B}\beta_2^2\right)}, \qquad \frac{A}{C} = \frac{-\left(\dfrac{A}{D} + \beta_2^2\right)\beta_2^2}{\delta_2^2\left(\dfrac{A^2C}{D^2G} + \beta_2^2\right)},$$

from which we obtain

$$\delta_1^2 = -\frac{\dfrac{B}{D}\left(\dfrac{A}{D}+\beta_2^2\right)}{\beta_2^2\left(1+\dfrac{E}{B}\,\beta_2^2\right)}, \qquad \delta_2^2 = -\frac{\beta_2^2\left(\dfrac{A}{D}+\beta_2^2\right)}{\dfrac{A}{C}\left(\beta_2^2+\dfrac{A^2G}{D^2C}\right)}; \qquad (22)$$

from the same equations we have also

$$\gamma_1^2 = -\frac{\dfrac{E^2}{BD}\left(\dfrac{A}{D}+\beta_2^2\right)}{1+\dfrac{E}{B}\beta_2^2}, \qquad \gamma_2^2 = -\frac{\dfrac{AG^2}{CD^2}\left(\dfrac{A}{D}+\beta_2^2\right)}{\beta_2^2+\dfrac{A^2G}{D^2C}}. \qquad (23)$$

From (20) we have again

$$\frac{D}{F} = \frac{\dfrac{A}{D}+\beta_2^2}{\delta_1^2\delta_2^2\left(\dfrac{AG}{DC}+\dfrac{E}{B}\beta_2^2\right)}, \qquad \delta_1^2\delta_2^2 = \frac{\dfrac{F}{D}\left(\dfrac{A}{D}+\beta_2^2\right)}{\dfrac{AG}{DC}+\dfrac{E}{B}\beta_2^2}. \qquad (24)$$

Combining (22) and (24) we have the quadratic equation determining β_2^2,

$$\beta_2^4 + \frac{AB}{ED}\left(\frac{GB+EC-FD-AH}{CB-AF}\right)\beta_2^2 + \frac{A^2GB}{D^2CE} = 0. \qquad (25)$$

β_2^2 being found from this equation, γ_1^2, γ_2^2, δ_1^2, δ_2^2 may be determined from equations (21), (22) and (23). We obtain no new parametric representation if we choose $\beta_2 = -\sqrt{\beta_2^2}$; in fact, substituting $-\rho_1$, $-\rho_2$ for ρ_1 and ρ_2 in (17) we obtain the same surface. If β_2^2 is negative, *i. e.*, $\beta_2 = \pm ik$, we still obtain real curves ρ_1 and ρ_2; in fact, it is easily seen from (21) that β_1, $\dfrac{\delta_1}{\gamma_1}$, $\dfrac{\delta_2}{\gamma_2}$ are also pure imaginaries. If then we put (17) in the form

$$\gamma_1^2 x = \frac{(\rho_1-\beta_1)(\rho_2-\beta_1)}{\left(\rho_1-\dfrac{\delta_1}{\gamma_1}\right)\left(\rho_2-\dfrac{\delta_1}{\gamma_1}\right)}, \qquad \gamma_2^2 y = \frac{(\rho_1-\beta_2)(\rho_2-\beta_2)}{\left(\rho_1-\dfrac{\delta_2}{\gamma_2}\right)\left(\rho_2-\dfrac{\delta_2}{\gamma_2}\right)}, \qquad z = \rho_1\rho_2,$$

and put $\rho_1 = i\rho_1'$, $\rho_2 = i\rho_2'$, the imaginary unit will disappear from the equations. We have therefore only to consider the two roots of (25), this equation being a quadratic in β_2^2. The discriminant is

$$\Delta = \frac{(GB+EC-FD-AH)^2}{(CB-AF)^2} - \frac{4EG}{CB}, \qquad (26)$$

and we have the three cases:

1°. $\Delta > 0$. *The roots are real and the curves ρ_1, ρ_2 are real twisted cubics.*

2°. $\Delta < 0$. *The roots are imaginary and the curves ρ_1 and ρ_2 are imaginary cubics;* it should be noticed that the surface is nevertheless real.

3°. $\Delta = 0$. *The roots are equal.* This limiting case we shall discuss later.

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The two roots of (25) *will give a double representation of the surface, and the second is obtained from the first precisely as before on pp.* 5 *and* 6. We found that, given a surface

$$x = \frac{(\rho_1 - \beta_1)(\rho_2 - \beta_1)}{(\gamma_1\rho_1 - \delta_1)(\gamma_1\rho_2 - \delta_1)}, \quad y = \frac{(\rho_1 - \beta_2)(\rho_2 - \beta_2)}{(\gamma_2\rho_1 - \delta_2)(\gamma_2\rho_2 - \delta_2)}, \quad z = \rho_1\rho_2, \quad (17)$$

we may obtain the same surface using the transformation

$$x_1 = \frac{AG}{BH} \cdot \frac{1}{x}, \quad y_1 = \frac{EA}{CH} \cdot \frac{1}{y}, \quad z_1 = \frac{AF}{DH} \cdot \frac{1}{z},$$

namely,

$$x = \frac{AG}{BH} \cdot \frac{(\gamma_1\rho_3 - \delta_1)(\gamma_1\rho_4 - \delta_1)}{(\rho_3 - \beta_1)(\rho_4 - \beta_1)}, \quad y = \frac{EA}{CH} \cdot \frac{(\gamma_2\rho_3 - \delta_2)(\gamma_2\rho_4 - \delta_2)}{(\rho_3 - \beta_2)(\rho_4 - \beta_2)},$$
$$z = \frac{AF}{DH} \cdot \frac{1}{\rho_3\rho_4}, \quad (17')$$

which we shall prove may be deduced from (17), if in this set of equations we substitute for β_1^2 the second root of (25); let us call it $\beta_1'^2$. We have then from (25)

$$\beta_2^2\beta_2'^2 = \frac{A^2GB}{D^2CE}, \quad (27)$$

and from the first of equations (21),

$$\beta_1^2\beta_1'^2 = \frac{A^2EC}{D^2BG}. \quad (28)$$

From the same equations and from (20) and (23) we have

$$\gamma_2^2\gamma_2'^2 = \frac{G^2AH}{D^2CE}, \quad \gamma_1^2\gamma_1'^2 = \frac{E^2AH}{D^2BG},$$
$$\delta^2\delta_1'^2 = \frac{BF}{GD}, \quad \delta_2^2\delta_2'^2 = \frac{CF}{ED}. \quad (29)$$

Introducing now in (17) the conjugate roots $\beta_1', \beta_2', \gamma_1', \gamma_2', \delta_1', \delta_2'$, we have the surface

$$x = \frac{(\rho_1 - \beta_1')(\rho_2 - \beta_1')}{(\gamma_1'\rho_1 - \delta_1')(\gamma_1'\rho_2 - \delta_1')}, \quad y = \frac{(\rho_1 - \beta_2')(\rho_2 - \beta_2')}{(\gamma_2'\rho_1 - \delta_2')(\gamma_2'\rho_2 - \delta_2')}, \quad z = \rho_1\rho_2. \quad (17'')$$

Substituting for $\beta_1', \beta_2', \gamma_1', \gamma_2', \delta_1', \delta_2'$ their values in terms of $\beta_1, \beta_2, \gamma_1, \gamma_2, \delta_1, \delta_2$, obtained from (27), (28) and (29), and putting

$$\rho_1 = \frac{\sqrt{\dfrac{FA}{DH}}}{\rho_3}, \quad \rho = \frac{\sqrt{\dfrac{FA}{DH}}}{\rho_4},$$

we obtain the surface (17') which was to be proven. We may now state the results obtained as follows:

THEOREM IV. *Given a cubic surface*

$$A + Bx + Cy + Dz + Exz + Fxy + Gyz + Hxyz = 0,$$

with the identical relation $AEGF = BCDH$ between the coefficients, it is always possible to find a parametric representation of the surface of the form

$$x = \frac{(\rho_1 - \beta_1)(\rho_2 - \beta_1)}{(\gamma_1\rho_1 - \delta_1)(\gamma_1\rho_2 - \delta_1)}, \quad y = \frac{(\rho_1 - \beta_2)(\rho_2 - \beta_1)}{(\gamma_2\rho_1 - \delta_2)(\gamma_2\rho_2 - \delta_2)}, \quad z = \rho_1\rho_2, \quad (17)$$

where $\beta_1^2, \beta_2^2, \gamma_1^2, \gamma_2^2, \delta_1^2, \delta_2^2$ are the roots of certain quadratic equations. A second mode of representation is obtained by taking for $\beta_1^2, \beta_2^2, \gamma_1^2, \gamma_2^2, \delta_1^2, \delta_2^2$ their respective conjugate values, so that the surface is also represented by the equations

$$x = \frac{AG}{BH} \frac{(\gamma_1\rho_3 - \delta_1)(\gamma_1\rho_4 - \delta_1)}{(\rho_3 - \beta_1)(\rho_4 - \beta_1)}, \quad y = \frac{EA}{CH} \frac{(\gamma_2\rho_3 - \delta_2)(\gamma_2\rho_4 - \delta_2)}{(\rho_3 - \beta_2)(\rho_4 - \beta_2)},$$

$$z = \frac{AF}{DH} \cdot \frac{1}{\rho_3\rho_4}. \quad (17')$$

The involutory transformation

$$x_1 = \frac{AG}{BH} \cdot \frac{1}{x}, \quad y_1 = \frac{EA}{CH} \cdot \frac{1}{y}, \quad z_1 = \frac{FA}{DH} \cdot \frac{1}{z}$$

transforms the curves $\rho_1 = C$, $\rho_2 = C$ into the curves $\rho_3 = C$, $\rho_4 = C$. These curves form two distinct pairs of families of the same species.

We shall now discuss the special case where $\Delta = 0$. *The two pairs of families are identical.* We shall prove that in this case *the surface has four double points;* that is, the surface is of the fourth class and is *a tetrahedral symmetrical surface.*

There are 15 right lines on the surface (17) in the finite part of space, the remaining 12 being the three edges of the tetrahedron in the plane at infinity, each counted 4 times, since they join the double points of the surfaces. Of these 15 lines 12 consist of 6 double lines. We shall prove that when $\Delta = 0$ these 6 pairs become three quadruple lines joining a fourth double point to the three already existing.

Putting $z = k_1$ in (18), the resulting conic will represent two straight lines, if the determinant

$$\begin{vmatrix} 0 & k_1 H + F & k_1 E + B \\ k_1 H + F & 0 & k_1 G + C \\ k_1 E + B & k_1 G + C & 2(k_1 D + A) \end{vmatrix} = 0,$$

which gives us the following values for k_1:

$$k_1 = -\frac{F}{H}, \quad (GE - HD)k_1^2 + (CE + BG - AH - FD)k_1 + BC - AF = 0. \quad (30)$$

In the same way, putting $y = k_2$, and $x = k_3$ in succession we find that k_2 and k_3 are determined by the following equations:

$$k_2 = -\frac{E}{H}, \quad (GF - CH)k_2^2 + (BG + DF - CE - AH)k_2 + BD - EH = 0, \quad (31)$$

$$k_3 = -\frac{G}{H}(EF - BH)k_3^2 + (CE + FD - BG - AH)k_3 + CD - AG = 0. \quad (32)$$

If the roots of the second of equations (30) are equal, we have

$$\Delta_1 = (EC + BG - AH - FD)^2 - 4(BC - AF)(GE - HD) = 0,$$

from which it follows that also

$$\Delta_2 = (BG + DF - CE - AH)^2 - 4(BD - EA)(GF - CH) = 0.$$

In fact, we have

$$C^2 E^2 + B^2 G^2 + A^2 H^2 + F^2 D^2 + 2ECBG - 2ECAH - 2ECFD - 2BGAH$$
$$- 2BGFD + 2AHFD = 4(BGCE + AFDH - AFGE - BCHD);$$

adding to each side of this equation $4CEAH - 4CEBG + 4GBDF - 4DFAH$
we have
$$(GB + DF - CE - AH)^2 = 4(BD - EA)(GF - CH).$$

Hence $\Delta_2 = 0$ whenever $\Delta_1 = 0$; in the same way it may be proved that $\Delta_3 = 0$.

But $\Delta_1 = \Delta$, since $GE - HD = \frac{BC}{EG}(CB - AF)$, so that *when* $\Delta \equiv 0$ *the three line-pairs degenerate into three double lines which all meet in a fourth double point whose coordinates are*

$$x_1 = \frac{CE + FD - BG - AH}{2(BH - EF)} = k_3,$$

$$y = \frac{BG + DF - CE - AH}{2(CH - GF)} = k_2, \quad\quad\quad (33)$$

$$z = \frac{EC + BG - AH - FD}{2(DH - EG)} = k_1.$$

We have thus proved the following

THEOREM V. *A cubic surface of the form*

$$A + Bx + Cy + Dz + Exz + Fxy + Gyz + Hxyz = 0, \qquad 9)$$

whose coefficients satisfy the relation

$$(GB + EC - FD - AH)^2 = 4(BC - AF)(GE - HD),$$

has 4 *double points; three of these are situated at the three vertices of the tetrahedron of reference in the plane at infinity, while the fourth is the point* $x = k_3$, $y = k_2$, $z = k_1$.

If the surface has a center of involution (*i. e.*, $AEGF = BCDH$), the coordinates of the fourth double point are:

$$x = \left(\frac{CD}{EF}\right)^{\frac{1}{2}}, \quad y = \left(\frac{BD}{GF}\right)^{\frac{1}{2}}, \quad z = -\left(\frac{BC}{EG}\right)^{\frac{1}{2}}. \qquad (34)$$

Transforming the origin to this point the surface is thrown into the well-known form

$$Pxy + Qxz + Ryz + Sxyz = 0, \qquad (35)$$

a surface belonging to the type known as a *tetrahedral symmetrical* surface.

It is also evident geometrically that if one line-pair of (9) degenerates into a double line the surface must have a fourth double point; in fact, each one is a double line, since it passes through one of the double points at infinity; hence, when they coincide we have a quadruple line, which means that there must be a fourth double point. The surface is therefore reducible to the form (35).

The most general form of a tetrahedral symmetrical surface is

$$Ax^m + By^m + Cz^m + D = 0, \qquad (36)$$

which, when $m = -1$ reduces to (35). We may therefore generalize our result as follows:

The transformation

$$x = x_1^m, \quad y = y_1^m, \quad z = z_1^m \qquad (37)$$

transforms the surface (9) into the form

$$A + Bx^m + Cy^m + Dz^m + Ex^m z^m + Fx^m y^m + Gy^m z^m + Hx^m y^m z^m = 0; \qquad (38)$$

which has the same characteristic properties as (9) viz.:

1°. *It contains a fourfold family of curves* ρ_1, ρ_2, ρ_3, ρ_4 *such that the pair* ρ_1 *and* ρ_2 *belong to the same species, and likewise* ρ_3 *and* ρ_4.

$2°$. The ∞^7 surfaces (38) remain invariant for the transformation

$$x = \lambda x_1, \quad y = \mu y_1, \quad z = \nu z_1.$$

$3°$. Any one of the surfaces (38) remains invariant for the involutory transformation

$$x^m = \frac{AG}{BH} \cdot \frac{1}{x_1^m}, \quad y^m = \frac{EA}{CH} \cdot \frac{1}{y_1^m}, \quad z^m = \frac{FA}{DH} \cdot \frac{1}{z_1^m}. \tag{37'}$$

$4°$. By this transformation the curves $\rho_1 = C$, $\rho_2 = C$ are transformed into $\rho_3 = C$, $\rho_4 = C$.

$5°$. It is always possible to find a parametric representation of the surface (38) of the form

$$x = \sqrt[m]{\frac{(\rho_1 - \beta_1)(\rho_2 - \beta_1)}{(\gamma_1\rho_1 - \delta_1)(\gamma_1\rho_2 - \delta_1)}}, \quad y = \sqrt[m]{\frac{(\rho_1 - \beta_2)(\rho_2 - \beta_2)}{(\gamma_2\rho_1 - \delta_2)(\gamma_2\rho_1 - \delta_2)}}, \quad z = \sqrt[m]{\rho_1\rho_2}, \tag{38'}$$

where β_1^2, β_2^2, γ_1^2, γ_2^2, δ_1^2, δ_2^2 are roots of certain quadratic equations. A second mode of representation is obtained by taking for β_1^2, ..., δ_2^2 their respective conjugate values, so that the same surface (38) is represented by the equations

$$x = \sqrt[m]{\frac{(\rho_3 - \beta_1')(\rho_4 - \beta_1')}{(\gamma_1'\rho_3 - \delta_1')(\gamma_1'\rho_4 - \delta_1')}}, \quad y = \sqrt[m]{\frac{(\rho_3 - \beta_2')(\rho_4 - \beta_2')}{(\gamma_2'\rho_3 - \delta_2')(\gamma_2'\rho_4 - \delta_2')}}, \quad z = \frac{1}{\sqrt[m]{\rho_3\rho_4}}. \tag{38''}$$

$\Delta = 0$ is the condition that these surfaces shall be tetrahedral symmetrical.

It should be noticed that or. all the surfaces (19) and (38) there exist two special asymptotic curves $\rho_1 = \rho_2$ and $\rho_3 = \rho_4$. When $\Delta = 0$ the asymtotic lines may be determined by quadratures according to a theorem proved by Lie.[*] Moreover, in this case there can be drawn through any point of the surface ∞^1 curves of the same species, so that, instead of a fourfold family, we obtain ∞^1 families of curves of the same species. In fact, if we transform the origin to the fourth double point the surface takes the form

$$Pxy + Qxz + Ryz + Sxyz = 0, \tag{39}$$

to which corresponds in the (X, Y, Z) space a translation-surface

$$Pe^{X+Y} + Qe^{X+Z} + Re^{Y+Z} + Se^{X+Y+Z} = 0,$$

or, putting $x = -x$, $y = -y$, $z = -z$,

$$Pe^Z + Qe^Y + Re^X + S = 0, \tag{40}$$

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[*] Lie-Scheffers, "Geometrie der Berührungstransformationen," p. 341.

a translation-surface connected with a degenerate quartic consisting of two intersecting conics; this translation-surface has, moreover,∞ [1] families of translation-curves, as was proved by Lie.[*] Since, therefore, (40) can be generated in ∞ [1] different ways, the surface (39) has ∞ [1] families of curves of the same species. We thus see that *the families of surfaces* (9) *and* (38) *are, from our standpoint, the most natural generalization of the tetrahedral symmetrical surfaces.*

III.

We proved above that $\Delta = 0$ is the condition that the surface (19) shall reduce to the form (39). It is well known that the dualistic of this surface is the so-called Steiner's surface of the fourth order and third class, viz.:

$$\sqrt{Ru} + \sqrt{Qv} + \sqrt{Pw} + \sqrt{S} = 0. \tag{39'}$$

If, therefore, we form the dualistic of (9), we sha 1 obtain the generalization of Steiner's surface. This new surface will be of the sixth order.

In order to find its equation we proceed as follows: We transform the origin to the point $\dfrac{-G}{H}, \ \dfrac{-E}{H}, \ \dfrac{-F}{H}$, so that the surface takes the form

$$A - \frac{GB}{H} - \frac{EC}{H} - \frac{FD}{H} + \frac{2EGF}{H^2} + \frac{BH - FE}{H}x + \frac{CH - GF}{H}y$$
$$+ \frac{DH - EG}{H}z + Hxyz = 0, \tag{40}$$

which we write

$$A_1 + B_1 x + C_1 y + D_1 z + Hxyz = 0.$$

The tangential coordinates are now

$$
\begin{aligned}
\rho u &= B_1 + Hyz, \\
\rho v &= C_1 + Hxz, \\
\rho w &= D_1 + Hxy, \\
\rho &= 3A_1 + 2B_1 x + 2C_1 y + 2D_1 z = A_1 - 2Hxyz,
\end{aligned} \tag{41}
$$

from which we have $Hxyz = \dfrac{A_1 - \rho}{2}$. From the first three equations (41) we get

$$\frac{H}{4}(A_1 - \rho)^2 = (u\rho - B_1)(v\rho - C_1)(w\rho - D_1); \tag{42}$$

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[*] See "Geometrie der Berührungstransformationen," p. 407.

from the equations (41) and

$$Hxyz = -(A_1 + B_1x + C_1y + D_1z) = \frac{A_1 - \rho}{2},$$

we get, after some reductions,

$$B_1(v\rho - C_1)(w\rho - D_1) + C_1(u\rho - B_1)(w\rho - D_1) + D_1(u\rho - B_1)(v\rho - C_1)$$
$$= \frac{H}{4}(A_1 - \rho)^2 - \frac{A_1 H}{2}(A_1 - \rho). \tag{43}$$

Equations (41) and (43) being respectively a cubic and a quadratic in ρ, we may eliminate ρ by Sylvester's dialytic method. We write the equations

$$a\rho^3 + b\rho^2 + c\rho + d = 0,$$
$$p\rho^2 + q\rho + r = 0, \tag{44}$$

where the coefficients have the following values:

$$a = uvw, \quad b = -\left(B_1 vw + C_1 uw + D_1 uv + \frac{H}{4}\right),$$

$$c = C_1 D_1 u + B_1 D_1 v + B_1 C_1 w + \frac{A_1 H}{2},$$

$$d = -\left(B_1 C_1 D_1 + \frac{A_1^2 H}{4}\right),$$

$$p = B_1 vw + C_1 uw + D_1 uv + \frac{H}{4}, \tag{45}$$

$$q = -2\left(C_1 B_1 w + D_1 B_1 v + C_1 D_1 u + \frac{A_1 H}{2}\right),$$

$$r = 3\left(B_1 C_1 D_1 + \frac{A_1^2 H}{4}\right),$$

which show that the following relations exist between the coefficients

$$p = -b, \quad q = -2c, \quad r = -3d.$$

Eliminating ρ from (44), we have

$$\begin{vmatrix} a & b & c & d & 0 \\ 0 & a & b & c & d \\ -b & -2c & -3d & 0 & 0 \\ 0 & -b & -2c & -3d & 0 \\ 0 & 0 & -b & -2c & -3d \end{vmatrix} = 0, \tag{46}$$

which, developed, gives us

$$27a^2d^2 - 18abcd + 4ac^3 - b^2c^2 + 4b^3d = 0,$$

a surface of the sixth degree in u, v, w.

If $d = 0$, the surface (46) reduces to

$$(4ac - b^2)c^2 = 0.$$

$c^2 = 0$ represents a double point, while the factor

$$4ac - b^2 = 0$$

represents *Steiner's surface.* In fact, if we write

$$d = 4(BH - FE)(CH - GF)(DH - EG) \\ + [(AH - GB - EC - FD)H + 2EGF]^2,$$

it will easily be seen to vanish whenever $\Delta = 0$. A little calculation will show that $d = H^2\Delta$. We get for Steiner's surface

$$4uvw\left(C_1D_1u + B_1D_1v + B_1C_1w + \frac{A_1H}{2}\right)$$
$$-\left(B_1uv + C_1uw + B_1vw + \frac{H}{4}\right)^3 = 0, \quad (47)$$

which is somewhat different in form from the one usually given, owing to the different method of obtaining the equation. (For the regular form see Salmon, "Geometry of Three Dimensions," p. 491, note.)

Returning to the surface (46), we shall prove that it has *one triple point, three double edges, and a cuspidal curve of order* 6.

We shall use Salmon's equations connecting the singularities of a surface, viz.:[*]

$$a'(n' - 2) = k' + \rho' + 2\sigma',$$
$$b'(n' - 2) = \rho' + 2\beta' + 3\gamma' + 3t',$$
$$c'(n' - 2) = 2\sigma' + 4\beta' + \gamma', \quad (48)$$
$$a' + 2b' + 3c' = n'(n' - 1),$$

where the letters have the meaning explained on pp. 580 and 581 of Salmon's treatise.

The points σ' are the intersections of an arbitrary plane with the curve UH, where U is the cubic surface and H the Hessian. Ordinarily there are 12 of these, but if we form the Hessian of the surface (19), we find that it intersects the cubic in a curve of the sixth order; hence $\sigma' = 6$. k' is the number of cuspidal edges on the tangent cone proper, and equals 9. a' is the class of a plane section of the cubic, hence $a' = 6$. γ' is zero, as always in the case of the dualistic of a cubic surface. Substituting these values in the above equations

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[*]Salmon, "Analytical Geometry of Three Dimensions," third ed., p. 580.

and putting $n' = 6$, we find $\rho' = 3$, $\beta' = 3$, $b' = 3$, $t' = 1$ and $c' = 6$, c' being the order of the cuspidal cubic, b' that of the double line, and t' the number of triple points. *When $\Delta = 0$ the cuspidal edge vanishes*; in fact, if we form the Hessian of a cubic surface with four double points, we find that it is tangent to the surface and does not intersect it. Hence $c' = 0$; the triple point and the double edges remain: the surface (19) has degenerated into a Steiner surface.

THEOREM VI. *The curves ρ_1, ρ_2, ρ_3, ρ_4 constitute a fourfold family of cubics on (46). These arrange themselves into two pairs which, when $\Delta = 0$, reduce to a single pair.*

Proof. The tangential coordinates of (18) are

$$\rho u = B + Ez + Fy + Hyz,$$
$$\rho v = C + Fx + Gz + Hxz,$$
$$\rho w = D + Ex + Gy + Hxy,$$
$$\rho = 2A + Bx + Cy + Dz - Hxyz.$$

Introducing in these equations the values of x, y, z from (17), we obtain the parametric representation of the dualistic surfaces and it is easily seen that the curves ρ_1 and ρ_2 are still twisted cubics; the same will also hold if we substitute the values of x, y and z obtained from equations (17''). When $\Delta = 0$, these two representations are identical; it should be noticed that these curves are not of the same species.*

The relation $AEGF = BCDH$ which we assumed to hold is a purely metrical one. Given any surface of the form

$$A + Bx + Cy + Dz + Exz + Fxy + Gyz + Hxyz = 0, \qquad (49)$$

we may by a simple translation throw it into a similar form and such that the new coefficients satisfy this relation; there exist ∞^2 such translations, since the coordinates of the new origin must satisfy one relation.

To find this we put $x = x_1 + \xi$, $y = y_1 + \eta$, $z = z_1 + \zeta$ in (49). The new equation may be written

$$A_1 + B_1 x_1 + C_1 y_1 + D_1 z_1 + E_1 x_1 z_1 + F_1 x_1 y_1 + G_1 y_1 z_1 + H_1 x_1 y_1 z_1, \qquad (49)$$

where the coefficients have the following values:

$$A_1 = A + B\xi + C\eta + D\zeta + E\xi\zeta + F\xi\eta + G\eta\zeta + H\xi\eta\zeta,$$
$$B_1 = B + E\zeta + F\eta + H\eta\zeta, \qquad C_1 = C + F\xi + G\zeta + H\xi\zeta,$$
$$D_1 = D + E\xi + G\eta + H\xi\eta, \qquad E_1 = E + H\eta,$$
$$F_1 = F + H\zeta, \qquad G_1 = G + H\xi, \qquad H_1 = H.$$

---

*On Steiner's surface the curves of the same species are quartic curves according to Lie. See Lie-Scheffers, " Geometrie der Berührungstransformationen," p. 333.

Forming the identity $A_1 E_1 G_1 F_1 = B_1 C_1 D_1 H_1$, we find that ξ, η, ζ must be a point on the cubic surface

$$T = AEGF - BCDH + (BEFG + AEFH - BDHF - ECBH)\xi$$
$$+ (EFGC + AFGH - CDFH - BGCH)\eta$$
$$+ (DEFG + AGEH - ECDH - BDGH)\zeta$$
$$+ (AFH^2 + EGF^2 - HDF^2 - CBH^2)\xi\eta$$
$$+ (AEH^2 + FGE^2 - DBH^2 - CHE^2)\xi\zeta$$
$$+ (EFG^2 + AGH^2 - BHG^2 - CDH^2)\eta\zeta$$
$$+ (AH^3 + 4HEGF + DFH^2 + CEH^2 - GBH^2)\xi\eta\zeta = 0.$$

If then ξ, η, ζ lies on this surface and if the values of these coordinates do not cause any of the coefficients A_1, B_1, ..., H_1 to vanish, the resulting transform of (49) *will have a center of involution.* By proceeding with (49') as we did with (18), the curves ρ_1, ρ_2, ρ_3 and ρ_4 may be obtained by solving a quadratic equation. Now any cubic surface having three double points may by a projective transformation be put in the form (49); hence the

THEOREM VII. *A cubic surface having three double points of which none are biplanar, may by a translation be put in the form*

$$A + Bx + Cy + Dz + Exz + Fxy + Gyz + Hxyz = 0,$$

where the coefficients satisfy the identical relation

$$AEGF = BCDH.$$

The center of involution, which is

$$\frac{\sqrt{AG}}{\sqrt{BH}}, \quad \frac{\sqrt{EA}}{\sqrt{CH}}, \quad \frac{\sqrt{FA}}{\sqrt{DH}},$$

may be chosen in ∞^2 ways. The surface having been thrown into the form (18), *the fourfold family of curves of the same species may be determined by solving a quadratic equation.*

The following example will show how the theory works. Let the surface be

$$3(xyz - 1) + 2(x - yz) + 2(y - xz) + 2(z - xy) = 0. \tag{50}$$

We have here,

$$AB = -6, \quad ED = -4, \quad GB = -4, \quad EC = -4, \quad FD = -4, \quad AH = -9,$$

and the equation (25), p. 9, becomes

$$\beta_2^4 - \frac{15}{4}\beta_2^2 + \frac{9}{4} = 0.$$

Solving, we find

$$\beta_2^2 = 3, \quad \beta_2'^2 = \frac{3}{4}.$$

Using the first value, we find, by substituting in equations (21) and (23),

$$\beta_1^2 = \frac{3}{4}, \quad \gamma_1^2 = \frac{3}{4}, \quad \delta_1^2 = \frac{1}{4},$$

$$\beta_2^2 = 3, \quad \gamma_2^2 = 3, \quad \delta_2^2 = 4.$$

Since $\dfrac{A}{D}$ is negative β_1 and β_2 must have like signs; thus, if we put $\beta_1 = \tfrac{1}{2}\sqrt{3}$,

we have $\beta_2 = \sqrt{3}$. $\dfrac{B}{E}$ being also negative and β_2 positive, it follows from the

second of equations (21) that δ_1 and γ_1 must have like signs. We put $\gamma_1 = \dfrac{\sqrt{3}}{2}$,

and $\delta_1 = \tfrac{1}{2}$. γ_2 and δ_2 must also have like signs; so we put $\gamma_2 = \sqrt{3}$ and $\delta_2 = 2$.
The parametric representation of the surface (50) is therefore

$$x = \frac{(\rho_1 - \tfrac{1}{2}\sqrt{3})(\rho_2 - \tfrac{1}{2}\sqrt{3})}{(\tfrac{1}{2}\sqrt{3}\rho_1 - \tfrac{1}{2})(\tfrac{1}{2}\sqrt{3}\rho_2 - \tfrac{1}{2})}, \quad y = \frac{(\rho_1 - \sqrt{3})(\rho_2 - \sqrt{3})}{(\sqrt{3}\rho_1 - 2)(\sqrt{3}\rho_2 - 2)}, \quad z = \rho_1\rho_2. \quad (50')$$

Suppose now we take the second value $\beta_1'^2 = \tfrac{3}{4}$; we find

$$\beta_1'^2 = 3, \quad \gamma_1'^2 = 3, \quad \delta_1'^2 = 4,$$

$$\beta_2'^2 = \tfrac{3}{4}, \quad \gamma_2'^2 = \tfrac{3}{4}, \quad \delta_2'^2 = \tfrac{1}{4},$$

Following the same rules for signs, we get the second representation

$$x = \frac{(\rho_3 - \sqrt{3})(\rho_4 - \sqrt{3})}{(\sqrt{3}\rho_3 - 2)(\sqrt{3}\rho_4 - 2)}, \quad y = \frac{(\rho_3 - \tfrac{1}{2}\sqrt{3})(\rho_4 - \tfrac{1}{2}\sqrt{3})}{(\tfrac{1}{2}\sqrt{3}\rho_3 - \tfrac{1}{2})(\tfrac{1}{2}\sqrt{3}\rho_4 - \tfrac{1}{2})}, \quad z = \rho_3\rho_4, \quad (50'')$$

which is identical with

$$x = \frac{(\tfrac{1}{2}\sqrt{3}\rho_3' - \tfrac{1}{2})(\tfrac{1}{2}\sqrt{3}\rho_4' - \tfrac{1}{2})}{\left(\rho_3' - \dfrac{\sqrt{3}}{2}\right)(\rho_4' - \tfrac{1}{2}\sqrt{3})}, \quad y = \frac{(\sqrt{3}\rho_3' - 2)(\sqrt{3}\rho_4' - 2)}{(\rho_3' - \sqrt{3})(\rho_4' - \sqrt{3})}, \quad z = \frac{1}{\rho_3'\rho_4'},$$

if we put $\dfrac{1}{\rho_3'} = \rho_3$ and $\dfrac{1}{\rho_4'} = \rho_4$. This surface has been modeled. (See plate.)

For the purpose of modeling it was found convenient to put $z = \dfrac{z'}{3}$ in (50),
so that the equation of the surface becomes

$$3xyz - 2yz - 2xz - 6xy + 6x + 6y + 2z - 9 = 0,$$

which has for parametric representation

$$x = \frac{(\rho_1 - \tfrac{1}{2}\sqrt{3})(\rho_2 - \tfrac{1}{2}\sqrt{3})}{(\tfrac{1}{2}\sqrt{3}\rho_1 - \tfrac{1}{2})(\tfrac{1}{2}\sqrt{3}\rho_2 - \tfrac{1}{2})}, \quad y = \frac{(\rho_1 - \sqrt{3})(\rho_2 - \sqrt{3})}{(\sqrt{3}\rho_1 - 2)(\sqrt{3}\rho_2 - 2)}, \quad z = 3\rho_1\rho_2;$$
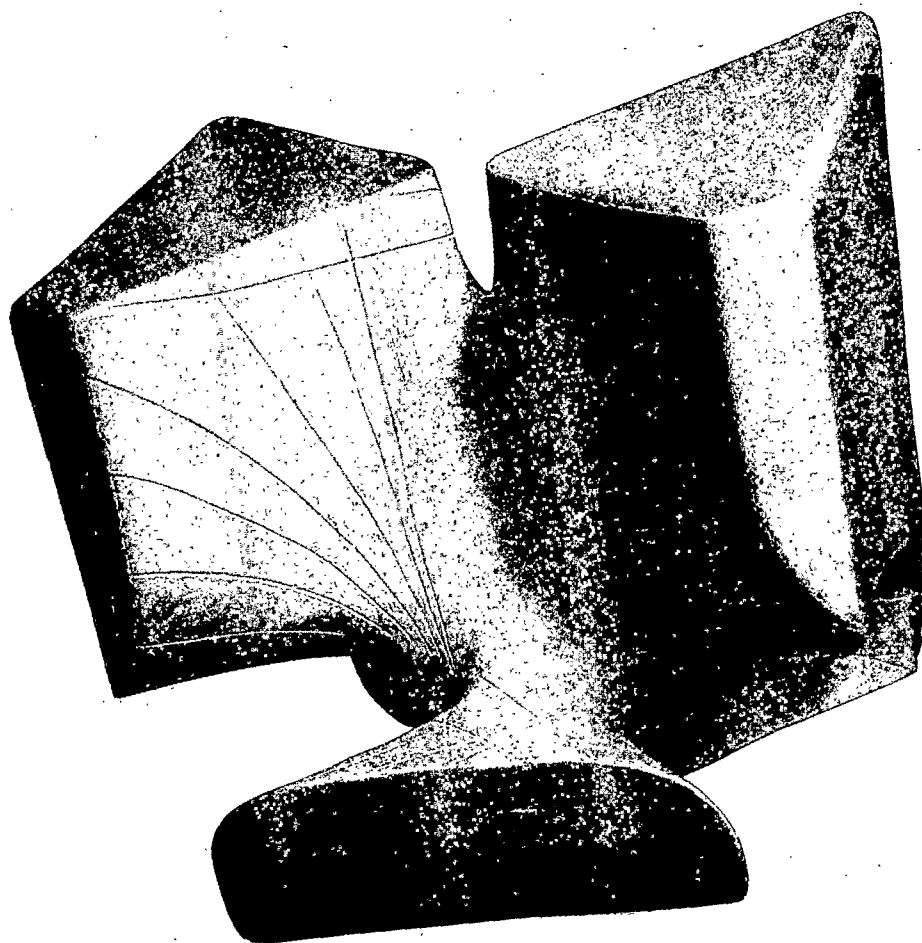
putting $\sqrt{3}\rho_1 = \rho_1'$, $\sqrt{3}\rho_2 = \rho_2'$, this may be written

$$x = \frac{4}{3} \cdot \frac{(\rho_1 - 3/2)(\rho_2 - 3/2)}{(\rho_1 - 1)(\rho_2 - 1)}, \quad y = \frac{1}{3} \cdot \frac{(\rho_1 - 3)(\rho_2 - 3)}{(\rho_1 - 2)(\rho_2 - 2)}, \quad z_1 = \rho_1 \rho_2.$$

The curves ρ_1, ρ_2, ρ_3, ρ_4 have been shown on the model.

IV.

It remains to consider the case where the unicursal quartic, which was the starting-point of the preceding theory, degenerates into a unicursal cubic and a straight line.[*]

Let the cubic have a double point. By means of a projective transformation it may be thrown into the form $y(1 - x^2) = x^3$, while the straight line may be written $y = mx + b$. The quartic now is

$$F(xy) = [y(1 - x^2) - x^3][y - mx - b] = 0, \tag{51}$$

from which we obtain

$$\begin{aligned} F'_{y_1} &= x_1^3 - (mx_1 + b)(1 - x_1^2) = (1 + m)x_1^3 + bx_1^2 - mx_1 - b, \\ F'_{y_2} &= -x_2^3 + (mx_2 + b)(1 - x_2^2) = -[(1 + m)x_2^3 + bx_2^2 - mx_2 - b]. \end{aligned} \tag{52}$$

Forming the Abelian integrals of the first kind, according to Lie's theorem,[†] we have the translation-surface

$$X = \int \frac{x_1 dx_1}{(x_1 - \alpha)(x_1 - \beta)(x_1 - \gamma)} - \int \frac{x_2 dx_2}{(x_2 - \alpha)(x_2 - \beta)(x_2 - \gamma)},$$

$$Y = \int \frac{x_1^3 dx_1}{(1 - x_1^2)(x_1 - \alpha)(x_1 - \beta)(x_1 - \gamma)} - \int \frac{x_2^3 dx_2}{(1 - x_2^2)(x_2 - \alpha)(x_2 - \beta)(x_2 - \gamma)}, \tag{53}$$

$$Z = \int \frac{dx_1}{(x_1 - \alpha)(x_1 - \beta)(x_1 - \gamma)} - \int \frac{dx_2}{(x_2 - \alpha)(x_2 - \beta)(x_2 - \gamma)},$$

where α, β, γ are the roots of the cubic equation

$$(1 + m)x^3 + bx^2 - mx - b = 0;$$

we shall assume these roots real and different, that is to say, the straight line

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[*] The theory of translation-surfaces connected, with a cubic and a straight line has been treated by Georg Wiegner in his thesis: *Ueber eine besondere Classe von Translationsflächen.* Inaugural dissertation. Leipzig, (1893). See also my paper in Am. Jour. of Math., Vol. XXIX, p. 370. Wiegner does not treat the case of unicursal cubics separately; in fact, the rôle that unicursality plays in this theory was not known to him, although very important, as is seen from a theorem proved by me in a paper published in Am. Jour. of Math., Vol. XXX, p. 179.

[†] See Lie-Scheffers, "Theorie der Berührungstransformationen," p. 411. A complete statement of the theorem is given in my paper, Am. Jour. of Math., Vol XXX, p. 171.

cuts the cubic in three distinct points. Transforming (53) by means of a linear projective transformation

$$X = Z', \quad Z = X', \quad mX - Y + bZ = Y',$$

we get a simpler form, viz.:

$$X' = \int \frac{dx_1}{(x_1 - \alpha)(x_1 - \beta)(x_1 - \gamma)} - \int \frac{dx_2}{(x_2 - \alpha)(x_2 - \beta)(x_2 - \gamma)},$$

$$Y' = \int \frac{dx_1}{1 - x_1^2},$$

$$Z' = \int \frac{x_1 dx_1}{(x_1 - \alpha)(x_1 - \beta)(x_1 - \gamma)} - \int \frac{x_2 dx_2}{(x_2 - \alpha)(x_2 - \beta)(x_2 - \gamma)}.$$

Integrating and putting $X'' = (\alpha - \beta)(\beta - \gamma)(\gamma - \alpha)X'$, $Y'' = 2Y'$ and $Z'' = (\alpha - \beta)(\beta - \gamma)(\gamma - \alpha)Z$, we obtain (dropping primes),

$$e^X = \frac{(x_1 - \alpha)^{\gamma - \beta}(x_1 - \beta)^{\alpha - \gamma}(x_1 - \gamma)^{\beta - \alpha}}{(x_2 - \alpha)^{\gamma - \beta}(x_2 - \beta)^{\alpha - \gamma}(x_2 - \gamma)^{\beta - \alpha}},$$

$$e^Y = \frac{1 + x_1}{1 - x_1}, \tag{54}$$

$$e^Z = \frac{(x_1 - \alpha)^{\alpha(\gamma - \beta)}(x_1 - \beta)^{\beta(\alpha - \gamma)}(x_1 - \gamma)^{\gamma(\beta - \alpha)}}{(x_2 - \alpha)^{\alpha(\gamma - \beta)}(x_2 - \beta)^{\beta(\alpha - \gamma)}(x_2 - \gamma)^{\gamma(\beta - \alpha)}},$$

By using a proper linear transformation of the coordinates X, Y and Z, which is not difficult to find, we may bring the surface (54) into the equivalent form

$$e^X = \frac{(x_1 - \beta)(x_2 - \gamma)}{(x_1 - \gamma)(x_2 - \beta)}, \quad e^Y = \frac{1 + x_1}{1 - x_1}, \quad e^Z = \frac{(x_1 - \alpha)(x_2 - \gamma)}{(x_1 - \gamma)(x_2 - \alpha)}. \tag{55}$$

Eliminating x_1 and x_2, we have the surface

$$(\alpha - \beta)(1 - \gamma)e^{X+Y+Z} + (1 + \gamma)(\beta - \alpha)e^{X+Z} + (1 - \beta)(\gamma - \alpha)e^{Y+Z}$$
$$+ (1 - \alpha)(\beta - \gamma)e^{X+Y} + (1 + \beta)(\alpha - \gamma)e^Z + (1 + \alpha)(\gamma - \beta)e^X = 0. \tag{55'}$$

Since α, β and γ depend on the two parameters m and b, we may say:

To a unicursal cubic and a straight line in the plane at infinity there correspond ∞^2 types of translation-surfaces of the form (55').

It now remains to find the second pair of translation-curves on (55'). Since now we can not fall back on the principle of symmetry, which holds only in the case of irreducible quartics, we have to begin *ab initio*. We write the quartic as before,

$$F(xy) = [y(1 - x^2) - x^3][y - mx - b] = 0,$$

but now we choose the two intersections of the variable line with $F(xy) = 0$ or the cubic, since before we took one point on the cubic and one on the straight line $y = mx + b$. We have

$$F'_{y_3} = x_3^3 - (mx_3 + b)(1 - x_3^2) = (1 + m)x_3^3 + bx_3^2 - mx_3 - b$$
$$F'_{y_4} = x_4^3 - (mx_4 + b)(1 - x_4^2) = (1 + m)x_4^3 + bx_4^2 - mx_4 - b.$$

Forming the Abelian integrals as before, we get

$$X = \int \frac{x_3 dx_3}{(x_3 - \alpha)(x_3 - \beta)(x_3 - \gamma)} \qquad + \int \frac{x_4 dx_4}{(x_4 - \alpha)(x_4 - \beta)(x_4 - \gamma)},$$
$$Y = \int \frac{x_3^3 dx_3}{(1 - x_3^2)(x_3 - \alpha)(x_3 - \beta)(x_3 - \gamma)} + \int \frac{x_4^3 dx_4}{(1 - x_4^2)(x_4 - \alpha)(x_4 - \beta)(x_4 - \gamma)}, \quad (56)$$
$$Z = \int \frac{dx_3}{(x_3 - \alpha)(x_3 - \beta)(x_3 - \gamma)} \qquad + \int \frac{dx_4}{(x_4 - \alpha)(x_4 - \beta)(x_4 - \gamma)},$$

which we shall transform, putting

$$X' = X, \quad Y' = -mX + Y - bZ, \quad Z' = Z,$$

so that (56) reduces to

$$X = \int \frac{x_3 dx_3}{(x_3 - \alpha)(x_3 - \beta)(x_3 - \gamma)} + \int \frac{x_4 dx_4}{(x_4 - \alpha)(x_4 - \beta)(x_4 - \gamma)},$$
$$Y = \int \frac{dx_3}{1 - x_3^2} \qquad\qquad + \int \frac{dx_4}{1 - x_4^2}, \qquad (56')$$
$$Z = \int \frac{dx_3}{(x_3 - \alpha)(x_3 - \beta)(x_3 - \gamma)} + \int \frac{dx_4}{(x_4 - \alpha)(x_4 - \beta)(x_4 - \gamma)}.$$

Integrating and transforming as before we have

$$e^X = [(x_3 - \alpha)(x_4 - \alpha)]^{\alpha(\gamma - \beta)}[(x_3 - \beta)(x_4 - \beta)]^{\beta(\alpha - \gamma)}[(x_3 - \gamma)(x_4 - \gamma)]^{\gamma(\beta - \alpha)},$$
$$e^Y = \frac{1 + x_3}{1 - x_3} \cdot \frac{1 + x_4}{1 - x_4}, \qquad\qquad (56'')$$
$$e^Z = [(x_3 - \alpha)(x_4 - \alpha)]^{\gamma - \beta}[(x_3 - \beta)(x_4 - \beta)]^{\alpha - \gamma}[(x_3 - \gamma)(x_4 - \gamma)]^{\beta - \alpha}.$$

Performing certain linear transformations on the variables X and Z and putting $Y = -Y$, we reduce (56'') to the form

$$e^X = \frac{(x_3 - \gamma)(x_4 - \gamma)}{(x_3 - \alpha)(x_4 - \alpha)}, \quad e^Y = \frac{1 - x_3}{1 + x_3} \cdot \frac{1 - x_4}{1 + x_4}, \quad e^Z = \frac{(x_3 - \gamma)(x_4 - \gamma)}{(x_3 - \beta)(x_4 - \beta)}. \quad (57)$$

Eliminating x_3 and x_4, we obtain the surface

$$(\beta - \alpha)(\alpha + 1)(\beta + 1)e^{X+Y+Z} + (\gamma - \beta)(\beta + 1)(\gamma + 1)e^{X+Y}$$
$$+ (\alpha - \beta)(\alpha + 1)(\gamma + 1)e^{Y+Z} \quad + (\alpha - \beta)(\alpha - 1)(\beta - 1)e^{X+Z} \qquad (57')$$
$$+ (\beta - \gamma)(\beta - 1)(\gamma - 1)e^X \quad + (\gamma - \alpha)(\alpha - 1)(\gamma - 1)e^Z = 0.$$

If now we translate this surface to a point ξ, η, ζ, it must be possible to determine these coordinates in such a way that the surface becomes identical with (55'). Putting therefore $X = X' + \xi$, $Y = Y' + \eta$, $Z = Z' + \zeta$ in (57') and putting the new coefficients equal to const. $\times$ the corresponding coefficients of (55'), we find, after easy calculations,

$$e^{\xi} = \frac{\gamma^2 - 1}{\beta^2 - 1}, \quad e^{\eta} = \frac{(1 + \alpha)(1 + \beta)(1 + \gamma)}{(1 - \gamma)(1 - \alpha)(1 - \beta)}, \quad e^{\zeta} = \frac{\alpha^2 - 1}{\gamma^2 - 1}.$$

The second mode of representation is therefore

$$e^X = \frac{\beta^2 - 1}{\gamma^2 - 1} \frac{(x_3 - \gamma)(x_4 - \gamma)}{(x_3 - \beta)(x_4 - \beta)}, \quad e^Y = \frac{(1+\alpha)(1+\beta)(1+\gamma)}{(1-\alpha)(1-\beta)(1-\gamma)} \frac{(1-x_3)(1-x_4)}{(1+x_3)(1+x_4)},$$

$$e^Z = \frac{\alpha^2 - 1}{\gamma^2 - 1} \frac{(x_3 - \gamma)(x_4 - \gamma)}{(x_3 - \alpha)(x_4 - \alpha)}. \tag{55'}$$

Hence the

THEOREM VIII. *To a cubic with a double point and a straight line there correspond ∞^2 types of translation-surfaces that can be generated in four different ways. These surfaces are of the form*

$$Ae^{X+Y+Z} + Be^{X+Z} + Ce^{Y+Z} + De^{X+Y} + Ee^X + Fe^Z = 0. \tag{60}$$

The converse will follow from a theorem which we shall prove later. If we transform (60), putting $X = -X$, $Y = -Y$, $Z = -Z$, we obtain a somewhat simpler form

$$A + Be^Y + Ce^X + De^Z + Ee^{Y-Z} + Fe^{X+Y} = 0; \tag{60'}$$

which is represented parametrically by the two sets of equations

$$e^X = \frac{(x_1 - \gamma)(x_2 - \beta)}{(x_1 - \beta)(x_2 - \gamma)}, \quad e^Y = \frac{1 - x_1}{1 + x_2}, \quad e^Z = \frac{(x_1 - \gamma)(x_2 - \alpha)}{(x_1 - \alpha)(x_2 - \gamma)}. \tag{60''}$$

$$e^X = \frac{\gamma^2 - 1}{\beta^2 - 1} \cdot \frac{(x_3 - \beta)(x_4 - \beta)}{(x_3 - \gamma)(x_4 - \gamma)}, \quad e^Y = \frac{(1-\alpha)(1-\beta)(1-\gamma)}{(1+\alpha)(1+\beta)(1+\gamma)} \cdot \frac{1+x_3}{1-x_3} \cdot \frac{1+x_4}{1-x_4},$$

$$e^Z = \frac{\gamma^2 - 1}{\alpha^2 - 1} \frac{(x_3 - \alpha)(x_4 - \alpha)}{(x_3 - \gamma)(x_4 - \gamma)}. \tag{60'''}$$

We shall now introduce the logarithmic transformation that we have used before, viz.:

$$e^X = x, \quad e^Y = y, \quad e^Z = z;$$

and we shall introduce the parameters ρ_1, ρ_2, ρ_3, ρ_4 instead of x_1, x_2, x_3, x_4 respectively. The surface (60) is thereby transformed into a cubic having three

double points, and passing through all four vertices of the tetrahedron and' through the edge $x = 0$, $z = 0$. This surface is

$$Axyz + Bxz + Cyz + Dxy + Ex + Fz = 0, \qquad (61)$$

which contains 4 families of curves that group themselves in pairs, each pair belonging to the same species; one pair, $\rho_1 = c$, $\rho_2 = c$, are plane conics; while the second pair, $\rho_3 = c$, $\rho_4 = c$, are twisted cubics. It should be noticed that now *there is no involutory transformation* (4) *which will transform* ρ_1 *and* ρ_2 *into* ρ_3 *and* ρ_4, *since the corresponding translation-surface has no center of symmetry.* The parametric equations are

$$x = \frac{\rho_1 - \beta}{\rho_1 - \gamma} \cdot \frac{\rho_2 - \gamma}{\rho_2 - \beta}, \quad y = \frac{1 + \rho_1}{1 - \rho_1}, \quad z = \frac{\rho_1 - \alpha}{\rho_1 - \gamma} \cdot \frac{\rho_2 - \gamma}{\rho_2 - \alpha}, \qquad (61')$$

or,

$$x = \frac{\beta^2 - 1}{\gamma^2 - 1} \cdot \frac{(\rho_3 - \gamma)(\rho_4 - \gamma)}{(\rho_3 - \beta)(\rho_4 - \beta)}, \quad y = \frac{(1+\alpha)(1+\beta)(1+\gamma)}{(1-\alpha)(1-\beta)(1-\gamma)} \cdot \frac{1 - \rho_3}{1 + \rho_3} \cdot \frac{1 - \rho_4}{1 + \rho_4},$$

$$z = \frac{\alpha^2 - 1}{\gamma^2 - 1} \cdot \frac{(\rho_3 - \gamma)(\rho_4 - \gamma)}{(\rho_3 - \alpha)(\rho_4 - \alpha)}. \qquad (61'')$$

If we use the second form (60'), we obtain the surface

$$A + By + Cx + Dz + Eyz + Fxy = 0, \qquad (62)$$

which may be obtained from (61) by the involutory transformation

$$x_1 = \frac{1}{x}, \quad y_1 = \frac{1}{y}, \quad z_1 = \frac{1}{z}.$$

This surface likewise has 4 sets of curves; *both pairs are conic sections, each pair being curves of the same species.* The parametric equations are

$$x = \frac{\rho_1 - \gamma}{\rho_1 - \beta} \cdot \frac{\rho_2 - \beta}{\rho_2 - \gamma}, \quad y = \frac{1 - \rho_1}{1 + \rho_1}, \quad z = \frac{\rho_1 - \gamma}{\rho_1 - \alpha} \cdot \frac{\rho_2 - \alpha}{\rho_2 - \gamma}, \qquad (62')$$

or

$$x = \frac{\gamma^2 - 1}{\beta^2 - 1} \cdot \frac{\rho_3 - \beta}{\rho_3 - \gamma} \cdot \frac{\rho_4 - \beta}{\rho_4 - \gamma}, \quad y = \frac{(1-\alpha)(1-\beta)(1-\gamma)}{(1+\alpha)(1+\beta)(1+\gamma)} \cdot \frac{1 + \rho_3}{1 - \rho_3} \cdot \frac{1 + \rho_4}{1 - \rho_4},$$

$$z = \frac{\gamma^2 - 1}{\alpha^2 - 1} \cdot \frac{\rho_3 - \alpha}{\rho_3 - \gamma} \cdot \frac{\rho_4 - \alpha}{\rho_4 - \gamma}. \qquad (62'')$$

We shall now prove the following

THEOREM IX. *Given any cubic surface of the form*

$$Axyz + Bxz + Cyz + Dxy + Ex + Fz = 0 \qquad (63)$$

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with non-vanishing coefficients, it is always possible to find a double parametric representation of the form (61') and (61'').

Proof. Let the surface (63) be transformed by means of the transformation

$$x = \lambda x_1, \quad y = \mu y_1, \quad z = \nu z_1.$$

The new surface,

$$A\lambda\mu\nu x_1 y_1 z_1 + B\lambda\nu x_1 z_1 + C\mu\nu y_1 z_1 + D\lambda\mu x_1 y_1 + E\lambda x_1 + F\nu z_1 = 0, \qquad (64)$$

will be identical with (61') if we put

$$\lambda\mu\nu A = (1-\gamma)(\alpha-\beta), \qquad \lambda\nu B = (1+\gamma)(\beta-\alpha),$$
$$\mu\nu C = (1-\beta)(\gamma-\alpha), \qquad \lambda\mu D = (1-\alpha)(\beta-\gamma), \qquad (65)$$
$$\lambda E = (1+\alpha)(\gamma-\beta), \qquad \nu F = (1+\beta)(\alpha-\gamma).$$

If these equations can be solved for λ, μ, ν, α, β, γ, we shall have a parametric representation of (63) by putting

$$x = \lambda \cdot \frac{(\rho_1-\gamma)(\rho_2-\beta)}{(\rho_1-\beta)(\rho_2-\gamma)}, \quad y = \mu \frac{1-\rho_1}{1+\rho_2}, \quad z = \nu \frac{(\rho_1-\gamma)(\rho_2-\alpha)}{(\rho_1-\alpha)(\rho_2-\gamma)},$$

or else

$$x = \lambda \cdot \frac{\beta^2-1}{\gamma^2-1} \cdot \frac{(\rho_3-\gamma)(\rho_4-\gamma)}{(\rho_3-\beta)(\rho_4-\beta)}, \quad y = \mu \frac{(1+\alpha)(1+\beta)(1+\gamma)}{(1-\alpha)(1-\beta)(1-\gamma)} \frac{1-\rho_3}{1+\rho_3} \cdot \frac{1-\rho_4}{1+\rho_4},$$

$$z = \nu \cdot \frac{\alpha^2-1}{\gamma^2-1} \cdot \frac{(\rho_3-\gamma)(\rho_4-\gamma)}{(\rho_3-\alpha)(\rho_4-\alpha)}.$$

We get from (65)

$$\mu\frac{A}{B} = \frac{\gamma-1}{\gamma+1}, \qquad \mu\frac{D}{E} = \frac{\alpha-1}{\alpha+1}, \qquad \mu\frac{C}{F} = \frac{\beta-1}{\beta+1},$$

$$\gamma = \frac{B+\mu A}{B-\mu A}, \qquad \beta = \frac{F+\mu C}{F-\mu C}, \qquad \alpha = \frac{E+\mu D}{E-\mu D}.$$

Substituting the values of α, β and γ in the two equations

$$\frac{\lambda A}{C} = \frac{1-\gamma}{1-\beta} \cdot \frac{\alpha-\beta}{\gamma-\alpha}, \qquad \frac{\gamma A}{D} = \frac{1-\gamma}{1-\alpha} \cdot \frac{\alpha-\beta}{\beta-\gamma},$$

obtained from (65), we get

$$\lambda = \frac{EC-DF}{BD-EA}, \qquad \nu = \frac{EC-DF}{AF-BC},$$

while μ is a root of the cubic equation

$$(F-\mu C)(B-\mu A)(E-\mu D) = 4\mu\frac{(ED-EA)(AF-BC)}{EC-DF},$$

obtained from the first of equations (65). Q. E. D. To the above theorem may
also be added

THEOREM X. *Given a quadric surface of the form*

$$A + By + Cx + Dz + Eyz + Fxy = 0 \qquad (62)$$

*with non-vanishing coefficients, it is always possible to find a double parametric
representation* (62') *and* (63''). *The two pairs of families of conic sections belong
each to the same species.*

The surfaces (62) bear therefore the same relation to Lie's quadrics (3) that the
cubic surfaces (61) do to the cubic surfaces (9). There is an essential difference
between the two categories: *There exists an involutory transformation that will
transform the surfaces* (3) *and* (9) *into themselves, while the surfaces* (61) *and* (62)
*do not admit of such a transformation, but are transformed into each other by the
same transformation.*

These theorems are also true, as regards the property in question, for
surfaces derived from (61) and (62) by means of the transformation

$$x = x_1^m, \quad y = y_1^m, \quad z = z_1^m;$$

that is, for the surfaces

$$Ax^m y^m z^m + Bx_1^m z_1^m + Cy_1^m z_1^m + Dx_1^m y_1^m + Ex_1^m + Fz_1^m = 0, \qquad (66)$$

$$A + By_1^m + Cx_1^m + Dz_1^m + Ey_1^m z_1^m + Fx_1^m y_1^m = 0, \qquad (67)$$

the parametric representations of which are easily deduced from equations
(55) and (57). There exists a one-to-one correspondence between the surfaces
(66) and (67) by virtue of the involutory transformation.

$$x = \frac{1}{x_1}, \quad y = \frac{1}{y_1}, \quad z = \frac{1}{z_1}.$$

To the four families of curves of the same species on (66) correspond 4 families
on (67); in fact, if in (66) we put $m = -m$ we obtain the surfaces (67).

If $\Delta = CE - DF = 0$, these surfaces also become tetrahedral symmetrical,
in which case there exist ∞^1 families of curves of the same species.

CONCLUSION.

The preceding investigations have thus revealed an extensive class of
surfaces on which there exist at least 4 families of curves of the same species.
In case the determinant Δ vanishes we have ∞^1 such families. The surfaces
naturally group themselves in three categories:

1°. Lie's quadrics:

$$Ayz + Bzx + Cxy + Lx + My + Nz = 0.$$

2°. The cubic surfaces:

$$A + Bx + Cy + Dz + Exz + Fxy + Gyz + Hxyx = 0.$$

3°. The cubic and quadric surfaces:

$$Axyz + Bxz + Cyz + Dxy + Ex + Fz = 0,$$
$$A + By + Cx + Dz + Eyz + Fxy = 0.$$

To these should also be added the surfaces obtained by transforming these three types by the transformation (37).

The first two categories admit of an involutory transformation

$$x = \frac{\lambda}{x_1}, \quad y = \frac{\mu}{y_1}, \quad z = \frac{\nu}{z_1}.$$

which leaves any given surface invariant. They have also a center of involution. The third category admits of no such transformation, and no center of involution exists. The determinant Δ has the following values:

1°. $\Delta_1 = (AL + BM - CN)^2 - 4LMAB.$

2°. $\Delta_2 = (GB + EC - FD - AH)^2 - 4(BC - AF)(GE - HD)$
  $(AEGF = BCDH).$

3°. $\Delta_3 = (DF - CE)^2.$

The vanishing of these invariants will be the necessary and sufficient conditions that the respective surfaces shall possess ∞^1 families of curves of the same species.

We note finally that any translation-surface of the form

$$f(e^x, e^y, e^z) = 0$$

gives rise to an algebraic surface having at least two families of curves of the same species. We note also all the algebraic surfaces conjugate to a tetrahedral complex. Such surfaces could be constructed if we knew how to find all the algebraic curves conjugate to such a complex, but only a few cases are known. (See S. Lie, "Berührungstransformationen," pp. 393, 394.) They contain two families of curves of the same species.

WEST VIRGINIA UNIVERSITY, *November* 1, 1909.

The Automorphic Transformations of the Binary Quartic.

By A. H. Wilson.

The transformations which are the subject of this paper have been discussed from several points of view. They form groups which are holohedrically isomorphic with certain substitution groups of four letters; and they have been given a picturesque geometric interpretation by use of the regular polyhedrons. When, however, the analytical expressions of these transformations have been obtained, it has always been by a specialization of the system of coordinates, which reduces the quartic to a canonical form. It is the object here to derive them for the general quartic.

1. *Substitution groups.* If, as usual, we let $[ABCD]$ denote a definite double ratio of four distinct points on a line, then the following permutations will in every case have the same value:

$$[ABCD], \quad [BADC], \quad [CDAB], \quad [DCBA].$$

If the four points have the harmonic position, so that $[ABCD] = -1$, there are eight permutations which have the same value:

$$[ABCD], \quad [BADC], \quad [CDAB], \quad [DCBA],$$
$$[BACD], \quad [ABDC], \quad [CDBA], \quad [DCAB].$$

For the equianharmonic position the letters may be permuted in twelve ways without changing the value of the double ratio:

$$[ABCD], \quad [BADC], \quad [CDAB], \quad [DCBA],$$
$$[BCAD], \quad [CBDA], \quad [ADBC], \quad [DACB],$$
$$[CABD], \quad [ACDB], \quad [BDCA], \quad [DBAC].$$

The three sets of permutations just given correspond to the substitution groups of four letters denoted, respectively, by G_4, the "Vierergruppe"; G_8, a dihedron group; and G_{12}, the tetrahedron group.

In the symbolism of these groups we have:

$$G_4: \quad 1, \quad (AB)(CD), \quad (AC)(BD), \quad (AD)(BC);$$

$$G_8: \quad 1, \quad (AB)(CD), \quad (AC)(BD), \quad (AD)(BC),$$
$$\quad (AB), \quad (CD), \quad (ACBD), \quad (ADBC);$$

$$G_{12}: \quad 1, \quad (AB)(CD), \quad (AC)(BD), \quad (AD)(BC),$$
$$\quad (ABC), \quad (ACD), \quad (BDC), \quad (ADB),$$
$$\quad (ACB), \quad (BCD), \quad (ABD), \quad (ADC).$$

If the substitutions of G_4 be denoted by $\iota_0, \iota_1, \iota_2, \iota_3$, respectively, then G_8 is obtained by adjoining to G_4 the transposition $\sigma = (AB)$, and G_{12} is obtained by adjoining to G_4 the circular substitution $\tau = (ABC)$. This composition of the groups will be useful in the following discussion:

$$G_4: \qquad \iota_0, \qquad \iota_1, \qquad \iota_2, \qquad \iota_3. \tag{1}$$

$$\left. \begin{array}{ccccc} G_8: & \iota_0, & \iota_1, & \iota_2, & \iota_3, \\ & \sigma\iota_0, & \sigma\iota_1, & \sigma\iota_2, & \sigma\iota_3. \end{array} \right\} \tag{2}$$

$$\left. \begin{array}{ccccc} G_{12}: & \iota_0, & \iota_1, & \iota_2, & \iota_3, \\ & \tau\iota_0, & \tau\iota_1, & \tau\iota_2, & \tau\iota_3, \\ & \tau^2\iota_0, & \tau^2\iota_1, & \tau^2\iota_2, & \tau^2\iota_3. \end{array} \right\} \tag{3}$$

2. *The decomposition of the quartic into factors.* The transformations are considered here as those which carry a binary quartic into a multiple of itself. In a paper published in the AMERICAN JOURNAL, Vol. XVII, page 185, Professor STUDY has discussed fully the irrational covariants of the quartic, and has incidentally derived the expressions of the transformations of G_4. His further results enable us to write the transformations of G_8 and G_{12}. For the sake of clearness I quote as much of the invariant theory of the quartic and of Professor STUDY's paper as is necessary for the purpose, adopting throughout the notation of the latter.

The rational covariant and invariant system of the quartic:

$$\left. \begin{array}{l} f = (ax)^4 = (a'x)^4 = \ldots = (a_1 x_2 - a_2 x_1)^4, \\ h = (hx)^4 = (h'x)^4 = \ldots = \tfrac{1}{2}(aa')^2 (ax)^2 (a'x)^2, \\ t = (tx)^6 = (t'x)^6 = \ldots = 2(ah)(ax)^3 (hx)^3, \\ g_2 = \tfrac{1}{2}(aa')^4, \qquad g_3 = \tfrac{1}{3}(ah)^4 = \tfrac{1}{6}(aa')^2 (aa'')^2 (a'a'')^2. \end{array} \right\} \tag{4}$$

The discriminant takes the form

$$G = \frac{g_2^3 - 27\, g_3^2}{16},\qquad (5)$$

and the cubic resolvent,

$$4\,e^3 - g_2\, e - g_3 = 0,\qquad (6)$$

with roots denoted by e_λ, e_μ, e_ν. The forms $h + e_\lambda f$, $h + e_\mu f$ and $h + e_\nu f$ are perfect squares, and they are factors of t^2. The quadratic factors of t are, however, given the following form for the sake of certain invariant properties:

$$
\left.
\begin{aligned}
l &= (lx)^2 = \frac{1}{s_\lambda}\sqrt{-h - e_\lambda f},\\[4pt]
m &= (mx)^2 = \frac{1}{s_\mu}\sqrt{-h - e_\mu f},\\[4pt]
n &= (nx)^2 = \frac{1}{s_\nu}\sqrt{-h - e_\nu f};
\end{aligned}
\right\} \qquad (7)
$$

where

$$
\left.
\begin{aligned}
s_\lambda &= -\sqrt{e_\nu - e_\lambda}\cdot\sqrt{e_\lambda - e_\mu},\\
s_\mu &= -\sqrt{e_\lambda - e_\mu}\cdot\sqrt{e_\mu - e_\nu},\\
s_\nu &= -\sqrt{e_\mu - e_\nu}\cdot\sqrt{e_\nu - e_\lambda}.
\end{aligned}
\right\} \qquad (8)
$$

Then is

$$\frac{t}{2} = -(e_\mu - e_\nu)(e_\nu - e_\lambda)(e_\lambda - e_\mu)\,lmn = -\sqrt{G}\cdot lmn.\qquad (9)$$

For the invariants and covariants of the quadratics, l, m, n, we have

$$
\left.
\begin{aligned}
(mn)_1 &= (mn)(mx)(nx) = -(lx)^2,\\
(nl)_1 &= (nl)(nx)(lx) = -(mx)^2,\\
(lm)_1 &= (lm)(lx)(mx) = -(nx)^2;
\end{aligned}
\right\} \qquad (10)
$$

$$
\left.
\begin{aligned}
\tfrac{1}{2}(ll')^2 &= \tfrac{1}{2}(mm')^2 = \tfrac{1}{2}(nn')^2 = 1,\\
(mn)^2 &= (nl)^2 = (lm)^2 = 0.
\end{aligned}
\right\} \qquad (11)
$$

The decomposition of f into quadratic factors is derived from equations (7), and in fact in three forms corresponding to the three ways of pairing the linear factors:

$$
\left.
\begin{aligned}
f &= -\frac{1}{e_\mu - e_\nu}(s_\mu m + s_\nu n)(s_\mu m - s_\nu n)\\[4pt]
&= -\frac{1}{e_\nu - e_\lambda}(s_\nu n + s_\lambda l)(s_\nu n - s_\lambda l)\\[4pt]
&= -\frac{1}{e_\lambda - e_\mu}(s_\lambda l + s_\mu m)(s_\lambda l - s_\mu m).
\end{aligned}
\right\} \qquad (12)
$$

The decomposition of f into linear factors is given as follows. We set

$$f = 4 (r_0 x)(r_1 x)(r_2 x)(r_3 x), \tag{13}$$

and have for the squares of these linear factors (see the paper cited, p. 210):

$$\left.\begin{aligned}
-2 (r_0 x)^2 &= \sqrt{e_\mu - e_\nu} \cdot l + \sqrt{e_\nu - e_\lambda} \cdot m + \sqrt{e_\lambda - e_\mu} \cdot n, \\
-2 (r_\lambda x)^2 &= \sqrt{e_\mu - e_\nu} \cdot l - \sqrt{e_\nu - e_\lambda} \cdot m - \sqrt{e_\lambda - e_\mu} \cdot n, \\
-2 (r_\mu x)^2 &= -\sqrt{e_\mu - e_\nu} \cdot l + \sqrt{e_\nu - e_\lambda} \cdot m - \sqrt{e_\lambda - e_\mu} \cdot n, \\
-2 (r_\nu x)^2 &= -\sqrt{e_\mu - e_\nu} \cdot l - \sqrt{e_\nu - e_\lambda} \cdot m + \sqrt{e_\lambda - e_\mu} \cdot n.
\end{aligned}\right\} \tag{14}$$

The simultaneous invariants of these forms are given by the formulæ

$$\left.\begin{aligned}
(r_0 r_\lambda)(r_\mu r_\nu) &= -(e_\mu - e_\nu); \quad (r_0 r_\lambda)^2 = (r_\mu r_\nu)^2 = e_\mu - e_\nu, \\
(r_\mu r_\nu)(r_\nu r_\lambda)(r_\lambda r_\mu) &= (r_\lambda r_0)(r_0 r_\mu)(r_\mu r_\lambda) = \sqrt[4]{G}, \\
(r_\nu r_\lambda)(r_\lambda r_0)(r_0 r_\nu) &= (r_0 r_\mu)(r_\mu r_\nu)(r_\nu r_0) = -\sqrt[4]{G}.
\end{aligned}\right\} \tag{15}$$

In the formulæ (14) the signs of the radicals may be arbitrarily chosen, while that of $\sqrt[4]{G}$ is then determined by

$$\sqrt[4]{G} = \sqrt{e_\mu - e_\nu} \cdot \sqrt{e_\nu - e_\lambda} \cdot \sqrt{e_\lambda - e_\mu}.$$

3. *Transformations expressed in symbolic notation.* The binary bilinear form in cogredient variables, $(dx)(\delta y)$, set equal to zero, will, if the discriminant does not vanish, establish a projective correspondence between the points of the binary domain. To a point x (that is, $x_1 : x_2$), there corresponds a point x', the vanishing point of $(dx)(\delta y) = (x'y) = 0$. The linear invariant of the form is $i = (d\delta)$, and the discriminant, $j = \frac{1}{2}(dd')(\delta\delta')$. The identity transformation is given by $(xy) = 0$. If $s_1 = (d_1 x)(\delta_1 y) = 0$ and $s_2 = (d_2 x)(\delta_2 y) = 0$ are two non-degenerate transformations, then their product is given by

$$s_1 s_2 = (d_1 x)(d_2 \delta_1)(\delta_2 y) = 0.$$

The condition that a transformation is involutory or of period 2, is $i = 0$; of period 3, $i^2 - j = 0$; of period 4, $i^2 - 2j = 0$; and so on.

4. *The transformations of the general case* are, as stated above, derived in the paper from which I have quoted. It is a well-known property of the covariant t of the quartic that the vanishing points of each of the quadratic factors are the double points of an involution, of which every form of the pencil $\varkappa f + \lambda h$ furnishes two pairs of points. In particular, to each of the three ways of pairing the points of f correspond in this manner, as double points of an involution, the root points of a quadratic factor of t.

The polar form $(ax)(ay)$, set equal to zero, is, if the discriminant $\frac{1}{2}(aa')$ does not vanish, a transformation pairing the points of the line according to an involution whose double points are given by the equation $(ax)^2 = 0$. We have in this way for the bilinear forms representing the transformations of the group leaving the general quartic unaltered:

$$G_4: \quad \iota_0 = (xy), \quad \iota_1 = (lx)(ly), \quad \iota_2 = (mx)(my), \quad \iota_3 = (nx)(ny). \tag{16}$$

5. *The transformations for the harmonic case* may be derived from the consideration of the quadratic factors of f, (12). If the points of f form a harmonic quadruple, the invariant g_3 vanishes, and conversely. The cubic resolvent (6) then becomes

$$4e^3 - g_2 e = 0,$$

with roots 0, $\frac{1}{2}\sqrt{g_2}$ and $-\frac{1}{2}\sqrt{g_2}$. According to the naming of the roots, we have different forms of the transformations. Setting

$$e_\lambda = 0, \quad e_\mu = \frac{1}{2}\sqrt{g_2}, \quad e_\nu = -\frac{1}{2}\sqrt{g_2},$$

the decomposition (12) becomes

$$f = \frac{1}{2}\sqrt{g_2}\,(l+m)(l-m)$$
$$= -\sqrt{g_2}\left(n + \frac{1}{\sqrt{-2}}\,l\right)\left(n - \frac{1}{\sqrt{-2}}\,l\right) \tag{17}$$
$$= \frac{1}{2}\sqrt{g_2}\,(l+\sqrt{-2}\,m)(l-\sqrt{-2}\,m);$$

where now

$$l = -\frac{2}{\sqrt{g_2}}\sqrt{-h}, $$
$$m = -\sqrt{\frac{2}{g_2}}\sqrt{-h - \frac{1}{2}\sqrt{g_2}\cdot f}, \tag{18}$$
$$n = -\sqrt{\frac{2}{g_2}}\sqrt{-h + \frac{1}{2}\sqrt{g_2}\cdot f}.$$

The second and third decompositions of f in (17) represent a pairing of the linear factors corresponding to the double ratio $\frac{1}{2}$ or 2. In the first this double ratio is -1, and from this we derive the desired transformations. The involutory transformation, namely, whose double points are given by $m + n = 0$, as well as that whose double points are given by $m - n = 0$, will interchange one pair

of the harmonic points, leaving the other points unaltered. For the group G_8 we have then, by (2), only to adjoin to G_4 the transformation

$$\sigma = (mx)\,(my) + (nx)\,(ny).$$

$G_8:$ $\iota_0 = (xy),$ $\iota_1 = (lx)\,(ly),$ $\iota_2 = (mx)\,(my),$ $\iota_3 = (nx)\,(ny),$

$\sigma\iota_0 = (mx)\,(my) + (nx)\,(ny),$ $\sigma\iota_1 = (mx)\,(my) - (nx)\,(ny),$ (19)

$-\sigma\iota_2 = (lx)\,(ly) + (xy),$ $-\sigma\iota_3 = -(lx)\,(ly) + (xy).$

The transformations $\sigma\iota_0$ and $\sigma\iota_1$ are involutory, the linear invariant i vanishing by virtue of (11); $\sigma\iota_2$ and $\sigma\iota_3$ are of period 4, as for each $i = -2$ and $j = 2$, and hence $\iota^2 - 2j = 0$.

5. *The transformations of the equianharmonic case* are derived from the decomposition of f into linear factors. We know that for this case there exist transformations which permute cyclically any three of the points, leaving the fourth unaltered.

We first derive a general formula for the transformation which takes the points x_1, x_2, x_3, into y_1, y_2, y_3.[*] This is given by the elimination of the coefficients $d_\alpha \delta_\beta$ from the equations

$$(dx)\,(\delta y) = 0, \quad (dx_1)\,(\delta y_1) = 0, \quad (dx_2)\,(\delta y_2) = 0, \quad (dx_3)\,(\delta y_3) = 0\,;$$

which gives as a result the transformation

$$(dx)\,(\delta y) = (x_1 x_3)\,(y_1 y_2)\,(x_2 x)\,(y_3 y) - (x_1 x_2)\,(y_1 y_3)\,(x_3 x)\,(y_2 y) = 0. \quad (20)$$

This is a statement of the equality of the double ratios $[x_1 x_2 x_3 x]$ and $[y_1 y_2 y_3 y]$, and as such may be given other equivalent forms. Setting now $y_1 = x_2$, $y_2 = x_3$, $y_3 = x_1$, we have for the *cyclic* transformation,

$$(dx)\,(\delta y) = (x_1 x_2)^2\,(x_3 x)\,(x_3 y) - (x_3 x_1)\,(x_2 x_3)\,(x_2 x)\,(x_1 y).$$

This may be given a symmetric form. Permuting cyclically the letters x_1, x_2, x_3 and adding the results, we obtain

$$3\,(dx)\,(\delta y) = (x_2 x_3)^2\,(x_1 x)\,(x_1 y) + (x_3 x_1)^2\,(x_2 x)\,(x_2 y) + (x_1 x_2)^2\,(x_3 x)\,(x_3 y)$$
$$- (x_3 x_1)\,(x_1 x_2)\,(x_3 x)\,(x_2 y) - (x_1 x_3)\,(x_2 x_3)\,(x_1 x)\,(x_3 y)$$
$$- (x_2 x_3)\,(x_3 x_1)\,(x_2 x)\,(x_1 y).$$

The last three terms may be easily reduced in pairs by use of the fundamental identity of binary forms. Thus, for example,

$$- (x_1 x_2)\,(x_2 x_3)\,(x_1 x)\,(x_3 y) - (x_2 x_3)\,(x_3 x_1)\,(x_2 x)\,(x_1 y)$$
$$= (x_1 x_2)^2\,(x_3 x)\,(x_3 y) + (x_2 x_3)\,(x_3 x_1)\,(x_1 x_2)\,.\,(xy).$$

---

[*] By x_1, x_2, x_3, y_1, y_2, y_3 are meant $x_{11} : x_{12}$, $x_{21} : x_{22}$, $x_{31} : x_{32}$, $y_{11} : y_{12}$, $y_{21} : y_{22}$, $y_{31} : y_{32}$.

Substituting these results, we have finally

$$
\left.
\begin{aligned}
2\,(dx)\,(\delta y) &= (x_2 x_3)^2\,(x_1 x)\,(x_1 y) + (x_3 x_1)^2\,(x_2 x)\,(x_2 y) \\
&\quad + (x_1 x_2)^2\,(x_3 x)\,(x_3 y) + (x_2 x_3)\,(x_3 x_1)\,(x_1 x_2)\,.\,(xy).
\end{aligned}
\right\} \quad (21)
$$

This is the general form of a transformation determined by the passing of x_1, x_2, x_3 into x_2, x_3, x_1, respectively.

Writing in this formula r_λ, r_μ, r_ν [see (13)] for x_1, x_2, x_3, we have the transformation sought:

$$
\left.
\begin{aligned}
2\,(dx)\,(\delta y) &= (r_\mu r_\nu)^2\,(r_\lambda x)\,(r_\lambda y) + (r_\nu r_\lambda)^2\,(r_\mu x)\,(r_\mu y) \\
&\quad + (r_\lambda r_\mu)^2\,(r_\nu x)\,(r_\nu y) + (r_\mu r_\nu)\,(r_\nu r_\lambda)\,(r_\lambda r_\mu)\,.\,(xy).^*
\end{aligned}
\right\} \quad (22)
$$

By use of (14) and (15) this may be written in terms of l, m, n, e_λ, e_μ, e_ν. From these formulæ we have

$$
\begin{aligned}
-\,2\,(r_0 r_\lambda)^2\,(r_\lambda x)\,(r_\lambda y) &= \\
(e_\mu - e_\nu)\,[\sqrt{e_\mu - e_\nu}\,.\,(lx)\,(ly) &- \sqrt{e_\nu - e_\lambda}\,.\,(mx)\,(my) - \sqrt{e_\lambda - e_\mu}\,.\,(nx)\,(ny)], \\
-\,2\,(r_0 r_\mu)^2\,(r_\mu x)\,(r_\mu y) &= \\
(e_\nu - e_\lambda)\,[-\sqrt{e_\mu - e_\nu}\,.\,(lx)\,(ly) &+ \sqrt{e_\nu - e_\lambda}\,.\,(mx)\,(my) - \sqrt{e_\lambda - e_\mu}\,.\,(nx)\,(ny)], \\
-\,2\,(r_0 r_\nu)^2\,(r_\nu x)\,(r_\nu y) &= \\
(e_\lambda - e_\mu)\,[-\sqrt{e_\mu - e_\nu}\,.\,(lx)\,(ly) &- \sqrt{e_\nu - e_\lambda}\,.\,(mx)\,(my) + \sqrt{e_\lambda - e_\mu}\,.\,(nx)\,(ny)], \\
-\,2\,(r_\mu r_\nu)\,(r_\nu r_\lambda)\,(r_\lambda r_\mu)\,.\,(xy) &= -\,2\,\sqrt{e_\mu - e_\nu}\,.\,\sqrt{e_\nu - e_\lambda}\,.\,\sqrt{e_\lambda - e_\mu}\,.\,(xy)\,;
\end{aligned}
$$

whence, by an easy reduction, (22) becomes

$$
\left.
\begin{aligned}
-\,2\,(dx)\,(\delta y) &= (\sqrt{e_\mu - e_\nu})^3\,.\,(lx)\,(ly) + (\sqrt{e_\nu - e_\lambda})^3\,.\,(mx)\,(my) \\
&\quad + (\sqrt{e_\lambda - e_\mu})^3\,.\,(nx)\,(ny) - \sqrt{e_\mu - e_\nu}\,.\,\sqrt{e_\nu - e_\lambda}\,.\,\sqrt{e_\lambda - e_\mu}\,.\,(xy).
\end{aligned}
\right\} \quad (23)
$$

The equianharmonic case is characterized by the vanishing of the invariant g_2. The cubic resolvent becomes then

$$
4\,e^3 - g_3 = 0,
$$

and its roots,

$$
e_\lambda = \varepsilon_1 \sqrt[3]{\frac{g_3}{4}}, \quad e_\mu = \varepsilon_2 \sqrt[3]{\frac{g_3}{4}}, \quad e_\nu = \varepsilon_3 \sqrt[3]{\frac{g_3}{4}}, \quad\quad (24)
$$

---

* This transformation will hold r_0 unaltered if $(r_0 r_\lambda)^4 + (r_0 r_\mu)^4 + (r_0 r_\nu)^4 = 0$; or by (14), if $(e_\mu - e_\nu)^2 + (e_\nu - e_\lambda)^2 + (e_\lambda - e_\mu)^2 = 0$. In the equianharmonic case [see (24)], this condition becomes $(\varepsilon_2 - \varepsilon_3)^2 + (\varepsilon_3 - \varepsilon_1)^2 + (\varepsilon_1 - \varepsilon_2)^2 = 0$, and this is a numerical identity.

where ε_1, ε_2, ε_3 are the cube roots of unity, named arbitrarily. The radical expressions in (23),

$$(\sqrt{e_\mu - e_\nu})^3, \quad (\sqrt{e_\nu - e_\lambda})^3, \quad (\sqrt{e_\lambda - e_\mu})^3, \quad \sqrt{e_\mu - e_\nu} \cdot \sqrt{e_\nu - e_\lambda} \cdot \sqrt{e_\lambda - e_\mu}, \quad (25)$$

take now, except for sign, the same value, and indeed each is equal to $\pm \frac{1}{2} \sqrt{(\sqrt{-3})^3 g_3}$, or each is equal to $\pm \frac{1}{2}\sqrt{-(\sqrt{-3})^3 g_3}$. From the transformation (23) we may omit then a common factor, the common value of the expressions (25). We shall have different transformations according to the choice of signs of the radicals in (14), which was arbitrary; there will in fact be exactly eight which are essentially different, each of which will transform the quartic into a multiple of itself. These may also be derived by (3); we have in fact, putting

$$\tau = -(lx)(ly) - (mx)(my) - (nx)(ny) + (xy),$$

$$G_{12}: \quad \iota_0 = (xy), \quad \iota_1 = (lx)(ly), \quad \iota_2 = (mx)(my), \quad \iota_3 = (nx)(ny),$$
$$\tau \iota_0 = -(lx)(ly) - (mx)(my) - (nx)(ny) + (xy),$$
$$\tau \iota_1 = (lx)(ly) - (mx)(my) + (nx)(ny) + (xy),$$
$$\tau \iota_2 = (lx)(ly) + (mx)(my) - (nx)(ny) + (xy),$$
$$\tau \iota_3 = -(lx)(ly) + (mx)(my) + (nx)(ny) + (xy), \quad\quad (26)$$
$$-\tfrac{1}{2}\tau^2 \iota_0 = (lx)(ly) + (mx)(my) + (nx)(ny) + (xy),$$
$$-\tfrac{1}{2}\tau^2 \iota_1 = -(lx)(ly) - (mx)(my) + (nx)(ny) + (xy),$$
$$-\tfrac{1}{2}\tau^2 \iota_2 = (lx)(ly) - (mx)(my) - (nx)(ny) + (xy),$$
$$-\tfrac{1}{2}\tau^2 \iota_3 = -(lx)(ly) + (mx)(my) - (nx)(ny) + (xy).$$

The respective invariants of the last eight of these transformations have the same value. Writing

$$(dx)(\delta y) = \pm (lx)(ly) \pm (mx)(my) \pm (nx)(ny) + (xy),$$

we have

$$i = (d\delta) = 2,$$
$$j = \tfrac{1}{2}(dd')(\delta\delta')$$
$$= \tfrac{1}{2}\{\pm (dl)(\delta l) \pm (dm)(\delta m) \pm (dn)(\delta n) + (d\delta)\}$$
$$= \tfrac{1}{2}\{(ll')^2 + (mm')^2 + (nn')^2 + 2\} = 4.$$

Hence, in every instance $i^2 - j = 0$; that is, the transformations $\tau \iota_\kappa$ are all of period 3.

HAVERFORD COLLEGE.

*Theorems on the Simple Finite Polygon and Polyhedron.**

By N. J. Lennes.

INTRODUCTION.

In his "Foundations of Geometry,"† Professor Hilbert defines the measure of area of a polygon as the sum of the measures of area of all the triangles into which a polygon may be decomposed by a definite decomposition. This definition implies :.

- 1. The existence of a criterion as to what constitutes a decomposition of a polygon into triangles.

2. That for every polygon there exists a finite decomposition; that is, a decomposition resulting in a finite number of triangles.

3. That any two decompositions into triangles of the same polygon result in sets of triangles such that the sum of the measures of area of the triangles is the same for both sets.

A similar definition of the measure of volume of a polyhedron requires similar propositions about it.

The proofs of the theorems here implied require as lemmas the theorems that the polygon and polyhedron separate the plane and the three-space respectively into two mutually exclusive sets, and these in turn are based upon the fundamental theorem that a straight line divides a plane into two such sets.

Hilbert proves‡ that the measure of area of a polygon as above defined is independent of any particular decomposition of the polygon. The corresponding theorem for the measure of volume of the polyhedron is proved by S. O. Schatunovsky.§ That a simple polygon separates the remaining points

---

...

in which it lies into two mutually exclusive sets has been proved by O. Veblen*
and also by Hans Hahn. † But so far as known to me there is in the literature
no proof that any polygon may be decomposed into a finite number of triangles.

The fundamental theorem on the polyhedron — viz., that it separates space
into two sets — has not been proved, nor has the theorem concerning its decom-
posability into tetrahedrons. Indeed it does not appear that a careful definition
of a polyhedron has been formulated.

The object of this paper is to formulate such a definition and to give proofs
of the theorems just enumerated. Two new proofs are given of the proposition
that a simple polygon separates the plane into two sets. The corresponding
theorem on the polyhedron may be proved in a manner analogous to either of
these, but only one of these proofs is here carried through.

The polygon is decomposed into triangles in such manner that no new
vertices are created; that is, every vertex of the resulting triangles is a vertex
of the original polygon. It is shown, however, that in general a polyhedron
can not be decomposed into tetrahedrons without creating new vertices.

In proving these theorems no use has been made of continuity, congruence
or the axiom on parallels.

Axioms I–IX of Professor Veblen ‡ or the axioms of Professor Hilbert,
excluding those just mentioned, are sufficient, as are the Projective Axioms of
Geometry given by Professor Moore. §

PART I. POLYGONS.

§ 1. *Preliminary Propositions.*

In this section are given a number of simple propositions that are required
in the argumentation that follows. Many of these theorems are proved by
Veblen. ‖ Each of the others follows by simple argumentation from those that
precede it. A number of these preliminary propositions are not needed before
we reach the polyhedron.

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* *Transactions of the American Mathematical Society,* Vol. V (1904), pp. 343–380.

† *Monatshefte für Mathematik und Physik,* Vol. XIX, pp. 289–303.

‡ *Loc. cit.*

§ *Transactions of the American Mathematical Society,* Vol. III (1903), p. 142.

‖ *Loc. cit.*

1. *For any set of n distinct points on a line a notation A_1, A_2, ..., A_n may be so arranged that the points A_i, A_j, A_k are in the order $A_i A_j A_k$ for all integral values of i, j, k such that $i < j < k$ and $i \geq 1$, $k \leq n$.*

DEFINITION. The points on a line between two of its points A, B constitute the *segment $A B$*. The points A and B are the *end-points* of the segment. The segment is said to *connect* its end-points.

2. *A point separates the remaining points of a line on which it lies into two sets of points such that a segment connecting points of the same set does not contain the point, while a segment connecting points of different sets does contain the point.*

3. *Any line of a plane separates the remaining points of the plane into two sets such that a segment connecting two points of the same set contains no point of the line, while a segment connecting points of different sets contains a point of the line.*

DEFINITIONS. The points of a line which lie on the same side of a point A of the line are called a *half-line* or *ray*. The ray is said to *proceed* from A. If B is any point of the ray, then the ray is referred to as the *ray $A B$* (not BA). A is the *end-point* of the ray. The two parts into which a line separates the remaining points of a plane are called *half-planes*.

A ray lying in the same line as the ray AB and proceeding from A but not containing B is the *complementary* ray of AB. Two non-complementary rays proceeding from the same point A together with the point A form an *angle* whose *vertex* is A. The two rays together with the vertex are sometimes referred to as the *boundary* of the angle. If the two rays are denoted by h and k, the angle formed by them is often denoted by $\angle (h, k)$.

4. *An angle separates the remaining points of the plane in which it lies into two sets, an interior and an exterior set, such that a segment connecting an interior and an exterior point contains one point of the angle, a segment connecting two interior points contains no point of the angle, while a segment connecting two exterior points and not containing the vertex contains two points or no point of the angle.*

5. *A segment connecting points one on each side of an angle lies entirely within the angle.*

6. *A ray proceeding from the vertex of an angle and containing a point within the angle lies entirely within it. If a segment connects points on different sides of the angle, it contains a point of this ray.*

7. *A ray proceeding from the vertex of an angle and lying within it forms with the sides of the angle two angles which have no interior point in common. Every point within the first angle lies within one of the new angles or lies on their*

common side, and every point within either of the new angles lies within the original angle.

8. *If a finite number of rays proceed from the vertex of an angle and lie within it, they may be ordered so as to form a series of angles no two of which have an interior point in common.*

PROOF. Let a segment connect points on different sides of the original angle. By (6) this segment meets each ray. Order the intersection points by (1) and apply (7).

DEFINITIONS. If the points A_1, A_2, A_3 are not collinear, the segments $A_1 A_2$, $A_2 A_3$, $A_3 A_1$, together with the points A_1, A_2, A_3, form a *triangle*. The segments are its *sides* and the points A_1, A_2, A_3 are its *vertices*. The points on the set of all segments whose extremities lie on different sides of a triangle constitute the *interior* points of the triangle. Points of the plane not on or within the triangle constitute its *exterior* points.

9. *A triangle so separates its interior and exterior points that a segment connecting an interior and an exterior point of the triangle meets it in one point, a segment connecting two interior points does not meet the triangle, while a segment connecting two exterior points and not containing a vertex meets the triangle in two points or in no point.*

Any number of rays proceeding from the same point P may be ordered by constructing a triangle of which P is an interior point and noting the order of the points on the triangle in which the rays meet it.

10. *The order of a set of distinct rays proceeding from a point is independent of the triangle used in ordering them.*

PROOF. Let a, b, c, d be half-lines proceeding from a point P. Suppose a does not make a straight line with any other of the half-lines. Then at least two of them, as b and c, lie on the same side of the line determined by a. Then one of the three half-lines a, b, c lies within the angle formed by the other two. Hence in this case the order is definite. If a and c form a straight line, then if b and d are on the same side of this line the proposition follows as before. If they are on opposite sides it likewise follows.

11. *A segment containing a point within an angle (or triangle) meets it in two points, or else at least one end-point of the segment lies on or within the angle (triangle).*

12. *If two triangles have an interior point in common, then one of the triangles is wholly or partly within the other.*

13. *A segment connecting a vertex of a triangle with a point in its opposite side*

lies within the triangle and forms with it two triangles which have no interior points in common. The interior points of the new triangles, together with their common side, are identical with the interior points of the original triangle.

14. The interior points of a triangle, and no other points, lie within each of the angles of the triangle.

15. If $[P]$* is a finite set of points within an angle BAC, and if B' is any point on the ray AB, then there is a point C' on the ray AC such that there is no point of $[P]$ on or within the triangle $AB'C'$.

PROOF. Draw segments from B' through each of the points of $[P]$ which are within the triangle $AB'C$ meeting AC in points P'_i. Then among the rays $B'P'_i$ there is one ray $B'P'_1$ such that no ray $B'P'_i$, and hence no point of $[P]$, lies within the angle $AB'P'_1$ (8). Let C' be any point of AP'_1. Then C' is a point such as is required (14), (13).

16. If $[\sigma]$ is any finite set of segments and ABC a triangle which has no point in common with $[\sigma]$ or its end-points, except possibly on BC, and if there are points of $[\sigma]$ within ABC, then there is an end-point P of a segment of $[\sigma]$ within ABC such that there is no point of $[\sigma]$ on or within the triangle ABP, and no end-point of $[\sigma]$ on it except the point P.

PROOF. By (11) there are end-points $[P]$ of segments of $[\sigma]$ within the triangle ABC. As in the proof of (15) find a point P_1 of $[P]$ such that there is no point of $[P]$ within the angle ABP_1. If P_1 is the only point of $[P]$ on the ray BP_1, then P_1 is the required point P. If not, let P_2 be that point of $[P]$ on the ray BP_1 which is nearest B. Then this is the required point P (14), (11).

DEFINITION. A point P is an *interior* point of a set of points $[P]$ if there exists a triangle t containing P as an interior point such that every other interior point of t is a point of $[P]$.

If every point of a set $[P]$ is an interior point of the set, then $[P]$ is said to be an *entirely open set.*

17. If $[P]$ is a set of points consisting of a finite set of segments $[\sigma]$ together with their end-points and any other finite set of points $[P]'$, then the remaining points of the plane constitute an entirely open set.

PROOF. Let Q be any point not in $[P]$ and let l be a line through Q not containing an end-point of a segment of $[\sigma]$. Then l contains only a finite

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* The notation $[P]$ is used to denote a set or class an element of which may be denoted by the symbol within the brackets.

number of points of $[P]$. Hence by (1) there is a segment AD on l containing Q but no point of $[P]$, A and D not being points of $[P]$. Through D pass a line l_1, distinct from l and not containing an end-point of $[\sigma]$. Then there is a segment $B'C'$ on l_1 containing D but no point of $[P]$. Then by (15) there are points B and C on $B'D$ and DC', respectively, such that no point of $[P]'$ and no end-point of a segment of $[\sigma]$ lies on or within either of the triangles ABD and ACD, and therefore neither on nor within the triangle ABC (13). Hence, by (11), there is no point of $[\sigma]$ on or within the triangle ABC, and therefore no point of $[P]$.

§ 2. *Separation of the Plane by a Polygon.*

DEFINITIONS. The segments $A_1 A_2$, $A_2 A_3$, $\ldots$, $A_{n-1} A_n$, together with the points A_1, A_2, $\ldots$, A_n, form a *broken line* $A_1 A_n$. The segments are the *sides* of the broken line and the points A_1, A_2, $\ldots$, A_n are its *vertices*. If no two sides have a point in common, no vertex lies on a side, and no two vertices coincide, except possibly A_1 and A_n, the broken line is said to be *simple*. If A_1 and A_n coincide, and if $A_1 A_n$ is a simple broken line, it is said to form a *simple polygon*.

If for any two points P_1 and P_2 of an entirely open set $[P]$ there exists a broken line $P_1 P_2$ lying in $[P]$, then $[P]$ is said to be *an open connected region*.

A point R is said to be *finitely accessible*, or simply *accessible*, from a point Q with respect to a set of points $[P]$, if there exists a broken line RQ containing no point of $[P]$, except possibly R or Q, or both. R and Q are also said to be *mutually accessible* with respect to $[P]$.

18. *An angle and a polygon have an even number of points in common provided that no vertex of either lies on the other.*

PROOF. In the polygon $A_1 A_2$, $A_2 A_3$, $\ldots$, $A_n A_1$ let A_1 be an exterior point of the angle (h, k). Following the sides of the polygon in the order given let A_i be the first vertex from A_1 which lies within the angle (h, k). Then the broken line $A_1 A_i$ meets the angle once (4). In this manner we can show that if A_j is the next vertex after A_i which lies outside the angle, then $A_i A_j$ meets the angle once, etc. Since the number of sides of the polygon is finite, this process leads to the starting-point A_1 after meeting the angle an even number of times. (Zero is here regarded as an even number.)

DEFINITION. A point P not on a polygon is said to be an interior point of the polygon if a ray proceeding from P and not containing a vertex of the

polygon meets it in an odd number of points. The point P is an exterior point of the polygon if such ray meets it in an even number of points.

That the external or internal character of the point as here defined depends *upon the point and the polygon only*, and not upon what particular ray is used, follows from (18). For if h and k are any rays proceeding from the same point and not containing vertices, it follows that h meets the polygon an odd or an even number of times according as k does.

If two points are both interior or both exterior they are said to lie on the *same side* of the polygon.

19. *If a broken line contains no point of a polygon, then all its points lie on the same side of the polygon.*

PROOF. Consider a broken line AB, BC, ..., KL which does not meet a certain polygon p. If either of the rays AB or BA contains no vertex of p, then by the definition of interior and exterior points all points of the segment AB, together with its end-points, lie on the same side of p. If each of these rays contains a vertex of p, proceed as follows:

From A', any point in the segment AB or the end-point A, construct a ray $A'K$ containing no vertex of p and hence not in the line AB. Then, by (1), (15) and (11), there is a point C on $A'K$, and not on p, such that there is no point of p on $A'C$ and BC, and such that the ray BC contains no vertex of p. Then B, C and A' are on the same side of p. Since A' is any point of AB, including A, it follows that every point of the segment AB, including the end-point A, is on the same side of p as the point B.

Continuing in this way, we prove that the whole broken line lies on the same side of p.

20. *A broken line connecting an interior and an exterior point of a polygon meets it at least once.*

21. *If a broken line connects two exterior or two interior points of a polygon, and if neither contains a vertex of the other, then they have an even number of points in common.*

22. *If a segment AB meets a polygon in only one point C which is not a vertex, then the segments AC and CB lie on opposite sides of the polygon.*

23. *A polygon has an exterior and also an interior set of points.*

24. *Every point of the plane not on a polygon is either internal or external to it.*

25. *If two points are on the same side of a polygon, they are mutually accessible with respect to the polygon.*

PROOF. Consider two exterior points A and B of a polygon p. Connect these with points C and D, respectively, on p, it being understood that there are no points of p on the segments AC and BD. About each vertex A_i of p construct a triangle t_i having the following properties: On or within each triangle there is no point of another of the triangles or of the polygon p except points on the segments $A_{i-1}A_i$, A_iA_{i+1} and the point A_i (17). The segments AC and BD also lie entirely outside each triangle. No triangle t_i has a vertex on the polygon.

Suppose that C lies on A_1A_2 of the polygon. Let Q be a point on A_1A_2 within t_2. Then, by (15), (11), there is a point R on AC such that RQ contains no point of p. Let RQ meet t_2 in S_2. Then A and S_2 are mutually accessible. In precisely the same manner we show there are points on t_2 and t_3, both exterior to p, which are mutually accessible. Denote these by S_2' and S_3 respectively. But S_2' and S_2 are mutually accessible with respect to p; for, tracing the triangle t_2 from S_2 to S_2', we meet p in two points or in no point (S_2 and S_2' both being exterior), and hence one way along t_2 from S_2 to S_2' fails to meet p. Hence A and S_3 are mutually accessible. Proceeding in this way we finally show that A and B are mutually accessible. In case A and B are both interior, the argument is precisely the same.

As a summary of (23), (24), (20), (25), (17) we now have:

26. *A simple polygon separates the remaining points of the plane in which it lies into two entirely open connected sets such that every broken line connecting points not in the same set meets the polygon.*

Theorem (26) has a certain form of converse as follows:

27. *If p consists of a finite set of segments [σ] together with their end-points, and if p separates the remaining points of the plane into two sets while no proper** subset of p does thus separate it, then p is a simple polygon.*

PROOF. (a) *Every end-point of a segment of p must be common to at least two segments.* For suppose in the segment A_1A_2 the point A_1 is an end-point of this segment only. Then if P and Q are not mutually accessible with respect to p, they are not mutually accessible with respect to a set consisting of p with the segment A_1A_2 removed; for if there exists a broken line connecting P and Q which contains no point of p except points of A_1A_2, then by suitable use of the process employed in the proof of (25) we can find a broken line PQ which goes around the end-point A_1, and which therefore contains no point of p.

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* A set is a proper subset of another set if it consists of some, but not all, the elements of that set.

(b) It follows that in p we may trace a broken line $A_1 A_2$, $A_2 A_3$, ...
without ever coming to a vertex from which we can not proceed without retracing
the segment last traced. Since the set of segments in p is finite, we must finally
reach a point in the broken line already traced. But this gives a simple polygon.
Since this polygon separates the plane, it follows that it contains every point
of p; for by hypothesis no proper subset of p separates the plane.

§ 3. *Decomposition of the Polygon.*

DEFINITION. A vertex A_k of a polygon is said to be *projecting* if there exists
a line which contains a point of each of the sides $A_{k-1} A_k$ and $A_k A_{k+1}$ and no
other point of the polygon.

28. *Every polygon has at least one projecting vertex.*

PROOF. Let p be the given polygon and $[\sigma]$ the set of all segments whose
end-points are vertices of p. Let l be a line meeting sides of the polygon but
containing none of its vertices. Let P be a point on l collinear with no two
vertices of p such that all intersection points of l with segments of $[\sigma]$ are on
the same side of P (1). From P construct rays through each of the vertices of p.
By means of (5) and (8) it follows that there are two of these rays, h and k, one
on each side of l, each of which forms an angle with l within which lies no vertex
of p and hence no point of p (11). Let h and k meet p in the vertices A_h and A_k
respectively. Then A_h and A_k are projecting vertices; for, let h' be the ray
next to h. Then any ray proceeding from P and lying within the angle (h', h)
meets p in the sides $A_{h-1} A_h$ and $A_h A_{h+1}$ and in no other points (6), (11).

29. *For any polygon there is a line such that the polygon lies entirely on the
same side of it.*

PROOF. In the construction used in the preceding proof extend the segment
$A_h A_k$ to a point Q and draw the line PQ. Now the ray PQ can not meet p,
since Q is outside the angle (h, k). Moreover, the point Q and the polygon p lie on
the same side of the line PA_h (with the exception of the point A_h, which lies on
the line). Hence that part of the line PQ which lies on the side of the line
PA_h opposite to Q can not meet p. That is, PQ is the required line.

DEFINITION. A set of triangles $[t]$ is said to constitute a decomposition of
a polygon p if: (a) no two triangles of $[t]$ have an interior point in common;
(b) every interior point of p lies on or within a triangle of $[t]$; (c) every
interior point of a triangle of $[t]$ is an interior point of p.

30. PROBLEM. *To decompose a given polygon into a set of triangles $[t]$ such that every vertex of $[t]$ is a vertex of the given polygon.*

SOLUTION. Let p be the given polygon and A_i a projecting vertex. Two cases are possible, viz.: (a) There may be vertices of p within the triangle $A_{i-1} A_i A_{i+1}$ or on the side $A_{i-1} A_{i+1}$ of this triangle; (b) there may be no vertex of p within the triangle $A_{i-1} A_i A_{i+1}$ or on $A_{i-1} A_{i+1}$.

In case (a), there is by (16), or by (1), a vertex A_h of p within $A_{i-1} A_i A_{i+1}$, or on $A_{i-1} A_{i+1}$, such that no vertex of p except A_h lies on or within the triangle $A_i A_{i+1} A_h$. Then p and the segments $A_i A_h$ and $A_{i+1} A_h$ form the triangle $A_i A_{i+1} A_h$ and one or two polygons, depending on whether $A_{i+1} A_h$ is a side of p or not. Denote, by $[\sigma]$, p together with the segments $A_i A_h$ and $A_{i+1} A_h$. By repeating the above process we obtain another triangle $A_j A_{j+1} A_k$ such that no point of $[\sigma]$ lies within it, and so on.

In case (b), A_{i-1}, A_i, A_{i+1} form a triangle such that no point of p lies within it or on the side $A_{i-1} A_{i+1}$. Then this last segment, together with p, forms a triangle and a polygon. In either case if the result is a triangle and one polygon, one new side is added, and this is common to the triangle and the polygon. Hence the new polygon has one side less than the original. If the result is a triangle and two polygons, two new segments are added and each of these is common to the triangle and one new polygon. Hence the two resulting polygons have together one side more than the original polygon. In any case this process will result in $n-2$ triangles, n being the number of sides of the polygon. Denote this set by $[t]$. Then:

(a) No two triangles of $[t]$ have an interior point in common (12).

(b) Every side of p is a side of exactly one triangle of $[t]$.

(c) Every side of a triangle of $[t]$ which is not a side of p is a side of exactly two triangles of $[t]$.

(d) Any two complementary* subsets of $[t]$ are such that there is at least one triangle in each set which has a side in common with a triangle of the other set. (Such sides are not sides of p.) (b), (c) and (d) follow directly from the construction.

(e) Any broken line which does not meet p lies entirely on or within triangles of $[t]$ if a single point of it lies on or within such triangle. This is

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* Two sets are said to be complementary subsets of a given set if, while having no common elements, the two sets together consist of the same elements as the given set.

obvious; for tracing such broken line across a side of one triangle leads into another, since each side which this broken line can meet is a common side of two triangles having no interior point in common (12).

(f) Any two points within triangles of $[t]$ or on sides of such triangles are mutually accessible with respect to p. This is an immediate consequence of (d).*

(g) Every interior point of p lies on or within a triangle of $[t]$. For by construction the interior points of the triangle $A_i A_{i+1} A_h$ are interior points of p. Hence, by (e) and (26), (g) follows.

It follows therefore that $[t]$ is a decomposition of p.

§ 4. *Another Proof that a Polygon Separates a Plane.*

We now assume the propositions of § 1; propositions (28), (29) and (a)–(f) under (30) of § 3; but not those of § 2. We refer to a set of triangles obtained from a given polygon as in (30), having the properties (a)–(f) there enumerated, as the set $[t]$ of that polygon.

31. *Any two points of a polygon are mutually accessible with respect to the polygon by means of a broken line lying on or within triangles of $[t]$.*

PROOF. Connect the given points with points within triangles of $[t]$ and apply (30, f).

32. *A point within a triangle of $[t]$ is not accessible with respect to p from a point not on or within such triangle* (30, e).

33. *Any point in the plane and any point on a polygon are mutually accessible with respect to the polygon.*

PROOF. If the given point lies on the polygon or on or within a triangle of $[t]$, this is proposition (30, f). If the point is exterior to every triangle of $[t]$, proceed as follows: From the point P obtained in the proof of (28) draw rays to each vertex of p. Order these rays and on each of them select points $B_1, B_2, \ldots, B_n$ such that all intersection points of the ray PB_i and the set $[\sigma]$ of (28) lie between P and B_i. Extend $A_h A_k$ of (28) to A' and A'', respectively. Then the broken line PA', $A'B_1$, B_1B_2, $\ldots$, B_{n-1}, B_n, $B_n A''$, $A''P$ forms a simple polygon p_1 which does not meet p. Connect two points A and B on PA' with points C and D on the same segment of p in such wise that the segments AC and BD do not meet. Then the polygon p, omitting the segment CD, and

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* Note that in § 3, up to this point, no use has been made of § 2.

the polygon p_1 omitting the segment AB, plus the segments AC and BD, form a simple polygon p_2. From this polygon we obtain a set of triangles $[t]_2$. It may readily be shown that the segment CD is not on or within one of these triangles, but that it is accessible from P. Hence, by (32) and (30, f), the triangles $[t]$ of the original polygon and the triangles of $[t]_2$ have no interior point in common. By (31) P is accessible to every point of p_2 and hence to every point of p. We now show that any point Q not in or on a triangle of $[t]$ is accessible from P. Construct a segment QR with R on a side $A_i A_{i+1}$ of p but no point of p on QR. Also connect P with R by means of a broken line $PP_1, \ldots, P_n R$ containing no point of p. Now QR and $P_n R$ do not lie within a triangle of $[t]$ and hence are on the same side of $A_i A_{i+1}$. Using (15), connect these segments by means of a segment not meeting p. Then we have a broken line connecting P and Q which contains no point of p. But every point of p is accessible from P and hence from Q.

DEFINITION. A line which does not meet a polygon, together with all points accessible from it with respect to the polygon and not on the polygon, are *exterior points* of the polygon. All the remaining points of the plane not on the polygon are *interior points*.

34. *Any point in a plane not on a given polygon is exterior to it or lies on or within a triangle of* $[t]$.

PROOF. Connect the given point Q with a point on p by means of a segment QR. If QR does not lie within a triangle of $[t]$, then by (33), (29) it is accessible from an exterior line and hence an exterior point. If QR has points within a triangle of $[t]$, then by (30, e) Q is on or within such triangle.

From the propositions (31)–(34) and (30, f) we have:

35. *A simple polygon separates the remaining points of the plane into two entirely open connected sets.*

PART II. POLYHEDRONS.

§ 5. *Definition of Polyhedron and Preliminary Propositions.*

DEFINITION. A *simple polyhedron*, or simply *polyhedron*, is a set of points consisting of a finite set $[t]$ of triangles, together with their interior points, having the following properties:

(1) Every side of a triangle of $[t]$ is common to an even number of triangles of this set.

(2) There exists no finite decomposition of the triangles of [*t*] into other triangles forming a set [*t*]' such that (1) is true of a proper subset of [*t*]'.

In this definition (2) is to be so understood that [*t*]' may be the same set as [*t*].

The vertices of the triangles are the *vertices* of the polyhedron. The sides of the triangles are the *edges* of the polyhedron, and the interior of any one triangle is one of the *faces* of the polyhedron.

We shall refer to the polyhedron as the polyhedron *p* or [*t*] as may be convenient.

It follows at once from the definition that no two triangles of a polyhedron lie in the same plane and have an interior point in common; for if this were the case, each triangle could be so decomposed that two triangles resulting from the decomposition would coincide, which is contrary to (2) of the definition.

For the purposes of this discussion the triangles of the polyhedron are regarded as decomposed so that (1) no two triangles have an interior point in common, (2) no edge of one triangle lies within another triangle. Such decompositions are always possible; for (1) if two triangles have an interior point in common, they have a segment in common, and this segment, extended if necessary, decomposes each of the triangles into two polygons, and these may in turn be decomposed into triangles. (2) If a side of one triangle lies within another, this side, extended if necessary, may be considered a decomposing segment.

The process of decomposing the triangles of the polyhedron in this manner may be called the normalizing process. The finitude of the process follows at once from the finitude of the number of triangles in the set we are considering.

36. *A plane separates space into two connected entirely open sets.*

37. *If two planes have one point in common, they have a second point in common and hence a line in common.*

38. *A plane has in common with a polyhedron at most a finite set of triangles with their interior points, and a finite set of segments, together with a finite set of points.*

That these include all of the polyhedron which can lie in the plane follows directly from the definition. Obviously any or all of these sets may be non-existent.

DEFINITIONS. That part of a plane which lies on one side of a line in it is called a *half-plane* and is said to proceed from this line.

Two non-coincident half-planes proceeding from the same line and not lying in the same plane form a *dihedral angle*. The two half-planes are called the *faces*

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of the angle, and the line from which they proceed, its *edge*. If the two faces are α and β, the angle is denoted by (α, β).

39. *A dihedral angle separates three-space into two connected entirely open sets.* One is called the interior and the other the exterior set.

40. *If A is a point in a face α of a dihedral angle (α, β), and if B is on the same side of the plane determined by α as the face β, then the segment AB lies wholly or partly within (α, β).*

§ 6.   The Separation of Three-Space by a Polyhedron.

41. *If $[t]'$ and $[t]''$ are complementary subsets of the set $[t]$, then there is at least one edge which is a side of an odd number of triangles in each set.*

PROOF. If every side of a triangle in $[t]'$ which is also a side of a triangle $[t]''$ is a side of an even number of triangles in $[t]'$, then every side of a triangle in $[t]'$ is a side of an even number of triangles in it, which is contrary to (2) of the definition.

42. *For any line l and any finite set of points $[P]$ not on l there is a plane through l not containing a point of $[P]$.*

43. *If a plane contains no three non-collinear vertices of a polyhedron, the plane has in common with it a finite set of segments with their end-points, together with a finite set of points.*

The finite set of points and also the set of segments may of course be non-existent.

44. *A plane which contains no vertex of a polyhedron has a finite set of simple polygons in common with it.*

PROOF. If the plane contains no point of the polyhedron, the theorem is verified, the number of polygons being zero.

If the plane contains points of the polyhedron, it must meet an edge, and the interior of each triangle of which this edge is a side has a segment in common with the plane (37). Hence the plane and the polyhedron have in common a set of segments such that every end-point is the end-point of an even number of segments. Starting with any one of these segments, we can trace a broken line until we return to some point in the line already traced. Since every vertex is the end-point of an even number of segments, this process may be repeated until the whole set of segments is exhausted, thus tracing a finite set of polygons.

45. *An angle not containing a vertex or a point in an edge of a polyhedron contains an even number of points in it, provided the vertex of the angle does not lie on the polyhedron.*

PROOF. (a) Suppose no vertex of the polyhedron lies in the plane of the angle. Then by (44) the polyhedron and the plane have a finite set of simple polygons in common; and since the angle does not contain a vertex of one of them (the angle contains no point of an edge of the polyhedron), then by (18) the angle contains an even number of points in each polygon and hence in the set of all of them.

(b) If the plane of the angle contains a vertex of the polyhedron, then by (42) obtain a point P such that the planes determined by the point P and each side of the given angle contain no such vertex. Let the given angle be (h_1, h_2) and let k be to the ray proceeding from the given point through P. Then we have two angles, (h_1, k) and (h_2, k), such that each contains an even number of points of the polyhedron. Hence if there is an even number of points on k, there is an even number of points on both h_1 and h_2, and hence on the angle (h_1, h_2). If there is an odd number of points of the polyhedron on k, there is an odd number of such points on h_1 and also h_2, and hence an even number on the angle (h_1, h_2).

DEFINITION. A point not on a polyhedron is *interior* or *exterior* to it according as a half-line proceeding from it and not meeting an edge or a vertex meets it in an odd or even number of points. Two points which are both exterior or both interior are said to lie on the same side of the polyhedron.

It follows from (45) that the interior and exterior quality of a point as here defined depends upon the point and the polyhedron and not upon the particular ray chosen, and also that every point not on the polyhedron is either interior or exterior.

46. *Any two points connected by a broken line which does not meet a polyhedron are both exterior or both interior to it.*

PROOF. Let AB be any segment of the broken line. If at least one of the rays AB and BA contains no vertex or point in an edge of the polyhedron, then by the definition the points of the segment AB, including its end-points, are all exterior or all interior.

If both rays AB and BA contain a vertex or a point of an edge, proceed as follows:

Let α be a plane containing the line AB but containing no vertex of the polygon except such as may lie on the line AB. Since the plane α contains no three non-collinear vertices of the polyhedron, the plane and the polyhedron have only a finite set of segments in common (43), and hence by (15) we can find a point D such that no point of the polyhedron lies within the triangle ABD. Within this triangle select any point C such that neither of the rays AC and BC contains a vertex or a point of an edge of the polyhedron. Then by the definition A and C are on the same side of the polyhedron, as are also B and C. Hence A and B are on the same side of it.

Proceeding in this manner, we can then show that the end-points of the broken line are on the same side of the polyhedron.

It is an immediate consequence of the preceding that:

47. *The points of a broken line which does not meet a polyhedron are all interior or all exterior points of the polyhedron.*

And also:

48. *A broken line connecting an interior and an exterior point meets the polyhedron.*

DEFINITION. Any two points connectible by a broken line exterior to the polyhedron have *external accessibility*, and if connectible by an interior broken line they have *internal accessibility*.

49. *Two points in the same face of a polyhedron or on the triangle enclosing this face have both internal and external accessibility.*

PROOF. Let A and B be the given points lying in a face t_1 of a polyhedron p. Denote by α the plane determined by t_1. Through A and B pass a plane β containing no vertex of p except such as may lie in the line AB. In β draw a line l through A meeting no edge or vertex of p. This is possible, since no edge lies in β except possibly in the line AB. Then by (43), (15) and (11) there is a point C on l such that no point of the segments AC and BC lies on p, and further such that the ray BC meets no edge or vertex of p. Then AC and BC and the point C are entirely interior or entirely exterior.

Similarly there is a point D on l in the order CAD such that AD and DB and D are entirely interior or entirely exterior. But if C is exterior, D is interior by the definition of interior and exterior. Hence A and B have both internal and external accessibility. If A is a vertex or lies in an edge, it may be connected with a point P in the face by passing a plane γ through AP distinct

from α, then ordering the segments in γ consisting of edges of the polyhedron or intersections with its faces, and applying (15) and (11).

50. *If B is a point of a face of a polyhedron and AB and CB are segments both entirely interior or both entirely exterior, then A and C are mutually accessible with respect to the polyhedron.*

PROOF. Let t_1 be the face of the polyhedron in which B lies. Then the segments AB and CB are readily shown to lie on the same side of the plane determined by t_1. In the plane of AB and CB apply successively (38), (15), (11) and (12), from which the theorem follows.

51. *If a segment AB meets a face of a polyhedron in a point C and meets the polyhedron in no other point, then AC is entirely interior and CB entirely exterior, or AC entirely exterior and BC entirely interior.*

PROOF. This is an immediate consequence of (50).

52. *If in a polyhedron all faces having a common side are divided into two sets, each containing an odd number of faces, then there are two points, one in a face of each set, which are internally accessible, and two points, one in each set, which are externally accessible.*

PROOF. Let AB be the common side and Q a point in it. Through Q pass a plane α not containing a vertex of the polyhedron p. Then the faces whose common side is AB will intersect α in a set of segments radiating from Q. Order these segments as in (10) and denote them by σ_i. In the plane α contruct a triangle t with Q as an interior point which does not meet p except in points on the segments σ_i (17). The triangle t is also to be such that the lines determined by its sides meet no vertex or edge of p and no vertex of the triangle lies on a segment σ_i.

Let the intersection points of t and σ_i be A_i. Suppose the segment (or broken line) $A_1 A_2$ is exterior; then $A_2 A_3$ is interior by the definition of interior and exterior. Similarly $A_3 A_4$ is exterior, etc.

If now the faces of which AB is a common edge are divided into two sets, the segments σ_i are divided into corresponding sets. Let one of these sets be σ_i'. Then at some point in the ordered sequence of segments σ_i an odd number of consecutive segments belong to σ_i', for otherwise σ_i' could not consist of an odd number of segments. Let the segment (or broken line) $A_{h-1} A_h$ which connects the first of these with a segment not of σ_i' be exterior; then, since these are alternately interior and exterior, it follows that the segment $A_k A_{k+1}$, which

connects the last of these segments with a segment not of $[\sigma]'$, is interior; and hence the theorem is proved.

53. *Any two points of a polyhedron have both internal and external mutual accessibility.*

Proof. Let A be any point on the polyhedron. It follows from (49) and (50) that if a single point of a face of the polyhedron is accessible from A, then every point of this face is thus accessible. Suppose not all points of the poly-hedron accessible from A. Let $[t]'$ be the set whose faces are thus accessible, and $[t]''$ its complementary subset of $[t]$. Then by (41) there is at least one edge which is a side of an odd number of triangles in each of these sets. Then by (52) points of some face of $[t]''$ are accessible with points of some face of $[t]'$ by both interior and exterior connection. Hence it follows that every point of the polyhedron is accessible from A by both interior and exterior connection.

54. *Any two interior points and also any two exterior points of a polyhedron are mutually accessible with respect to the polyhedron.*

Proof. Let A and B be any two exterior points of a polyhedron p. Connect them to points in faces of the polyhedron by segments AC and BD such that the rays AC and BD do not contain a vertex or a point of an edge of p, and the segments AC and BD contain no points of p. By (53) C and D are mutually accessible by exterior connection, and hence by (50) A and B are thus accessible.

If the given points are both interior, the argument is precisely the same.

From (48), (54), and from the fact that every point not on a polyhedron is either interior or exterior to it, we now have:

55. *A polyhedron separates the remaining points of a three-space into two connected sets.*

It is also easy to show that these sets are both entirely open, the definition of entirely open set being modified to suit the case of three-dimensional space.

56. *For every polyhedron there is a complete line which is entirely exterior to it, while no line is entirely interior to it.*

Proof. This follows at once from (29) by passing a plane through the polyhedron.

57. *If an interior point of a polyhedron lies within a dihedral angle, then a point of the polyhedron lies within the dihedral angle.*

PROOF. Let l be the edge of the dihedral angle and P the given point. Connect P with a point A on l exterior to the polyhedron (56). Then the segment AP lies entirely within the dihedral angle and meets the polyhedron (54).

§ 7. Decomposition of the Polyhedron.

DEFINITION. A finite set $[p]$ of polyhedrons is said to constitute a *decomposition* of a polyhedron p if:

(a) No two polyhedrons of $[p]$ have an interior point in common.

(b) Every interior point of a polyhedron of $[p]$ is an interior point of p.

(c) Every interior point of p is an interior point of a polyhedron of $[p]$ or lies on one of these polyhedrons.

58. *If $[p]$ is a decomposition of a polyhedron p, and if P is a point on a polyhedron p_1 of $[p]$, then P is an interior point of p cr lies on this polyhedron.*

PROOF. Let A be any interior point of p_1. Connect A and P by a broken line which lies entirely within p_1 (55). Then by the definition of decomposition the broken line AP lies within p and hence contains no point of p. Therefore P can not be an exterior point of p, since every broken line connecting an interior and exterior point meets the polygon (48).

59. *Not every polyhedron can be decomposed into tetrahedrons in such manner that every vertex of the tetrahedrons is a vertex of the polyhedron.*

PROOF. This proposition is proved by exhibiting a polyhedron of which no such decomposition exists.

Let E_1, C, E_2 (in Fig. 1) be any three non-collinear points. S_1 and S_2 are points on CE_1 and CE_2 respectively in the orders CE_1S_1 and CE_2S_2. Let O be the intersection point of the segments S_1E_2 and S_2E_1. (That these segments meet is an immediate consequence of the triangle transversal axiom.) V_1 and V_2 are points on the segments OS_1 and OS_2 respectively. Connect S_1S_2. Let the intersections of S_1S_2 with the lines CV_1 and CV_2 be O_1 and O_2 respectively. It follows that we have the orders CV_1O_1 and CV_2O_2. A is any point not in the plane E_1CE_2. Connect O, O_1 and O_2 with A. On segments AO_1 and AO_2 select points D_1 and D_2 respectively, and on the line AO select B in the order AOB. Then we consider the following points, triangles and segments:

Seven points:
$$\left\{ \begin{array}{l} D_1, \; E_1, \\ A, \; B, \; C, \; D_2, \; E_2. \end{array} \right.$$

Ten triangles:
$$\left\{ \begin{array}{l} ABD_1, \; ACD_1, \; CD_1E_1, \; BD_1E_1, \; BE_1E_2, \\ ABD_2, \; ACD_2, \; CD_2E_2, \; BD_2E_2, \; CE_1E_2. \end{array} \right.$$

Fifteen segments:
$$\left\{ \begin{array}{l} AB, \quad BD_1, \quad CD_1, \\ AC, \quad BD_2, \quad CD_2, \quad D_1E_1, \\ \qquad\qquad\qquad\qquad\qquad\qquad\; E_1E_2, \\ AD_1, \quad BE_1, \quad CE_1, \quad D_2E_2, \\ AD_2, \quad BE_2, \quad CE_2. \end{array} \right.$$

These triangles form a polyhedron; for:

(1) Every side of a triangle is common to exactly two triangles.

(2) No subset of these triangles exists such that (1) of the definition of a polyhedron holds. This remains true for any finite decomposition of the triangles here given, for every such decomposition leaves every edge common to exactly two triangles.

We now show that any triangle not a triangle of this polyhedron but determined by three of these points lies wholly or partly outside the polyhedron. This is best done by showing that every segment not an edge of this polyhedron but determined by two of its vertices lies wholly or partly outside the polyhedron. These segments are:

$$\left\{ \begin{array}{l} AE_1, \; D_1E_2, \\ \qquad\qquad\quad BC, \; D_1D_2, \\ AE_2, \; D_2E_1. \end{array} \right.$$

All vertices of the polyhedron except A, C, E_1 lie on the same side of the plane determined by these points. Hence there is no interior point of the polyhedron in this plane, and the only points of the polyhedron in this plane are A, C, E_1 and those of the segments AC and CE_1. Hence the segment AE_1 is entirely exterior to the polyhedron. In the same manner AE_2, BC and D_1D_2 may be shown to be entirely exterior to the polyhedron.

We now notice that there are no points of the polyhedron within the dihedral angle whose edge is AO and whose faces contain the points O_1 and O_2. Hence by (57) there is no interior point of the polyhedron within this dihedral angle. Further, E_1 and D_1 lie on the same side of the plane AOD_2. Hence by (40) part of the segment D_2E_1 lies partly with the dihedral angle O_1AOO_2. But

this part of D_2E_1 is exterior to the polyhedron. In the same manner we may show that D_1E_2 is partly exterior to the polyhedron.

We now observe that every triangle determined by three of the points A, B, C, D_1, D_2, E_1, E_2 which is not a triangle of the given polyhedron has one of these segments as a side. Hence by (58) no such triangle can belong to a set of triangles which constitute a set of polyhedrons that form a decomposition of the given polyhedron.

DEFINITION. A polyhedron is *convex* if all its interior points lie on the same side of every plane determined by its faces.
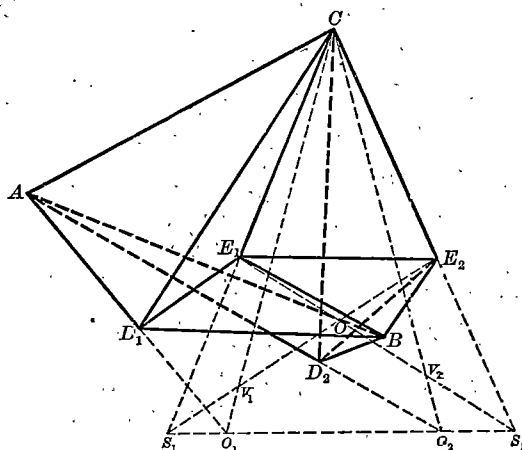
FIG. I.

60. *A ray proceeding from a vertex A of a tetrahedron and containing an interior point P meets the face opposite the given vertex in a point B in the order APB.*

61. PROBLEM. *To decompose any convex polyhedron into a set of tetrahedrons.*

SOLUTION. Let A be any vertex of a given convex polyhedron p. Consider all triangles of p which do not lie in the same plane with a triangle of p of which A is a vertex. Any one of these triangles, together with the three triangles determined by its sides and the point A, form a tetrahedron. The set $[\tau]$ of all such tetrahedrons form a decomposition of p; for:

(a) *No two of these tetrahedrons have an interior point in common.*

Suppose there is such common interior point P. Then the ray AP must meet p in two different faces (60). Since no two faces have a point in common, it would follow by (53) that not all interior points of the polyhedron lie on the same side of each of the planes determined by these faces.

8

(b) *Every interior point of p lies on or within a tetrahedron of* $[\tau]$.

Let P be any interior point of p. Then the ray AP meets p in a point not in a plane with triangles of which A is a vertex, and hence it meets a face, an edge or vertex of a tetrahedron of $[\tau]$. That is, P is on or within one of these tetrahedrons.

(c) *Every point within a tetrahedron of* $[\tau]$ *lies within p.*

Let P be any point within a tetrahedron of $[\tau]$. Then the ray AP meets a face of p in a point B in the order APB (60), and in no other point. Hence, by the definition of interior and exterior points P is an interior point of p.

62. *If a plane α contains an interior point of a polyhedron p, then there exists a set of triangles,* $[t]_{p,\alpha}$, *such that every interior point of p which lies in α is on or within a triangle of* $[t]_{p,\alpha}$, *while no exterior point lies within one of these triangles.*

Proof. If α contains a face of p, then an interior point of p can be connected with a point in this face only by crossing a vertex or edge of p (9), provided the connecting broken line lies in α. Denote by $[\sigma]$ the set of segments, and their end-points, consisting of all edges of p in α and all intersections of α with faces of p.

Let P be any interior point of p in α and $[P]$ all interior points of p in α accessible from P with respect to $[\sigma]$. By (56) this set does not include the whole plane. Hence there is a subset of $[\sigma]$ which fulfils the conditions of (27), and hence forms a simple polygon p_1. All points of this polygon are points of p, and all its interior points are interior points of p.

If there is an interior point P_1 of p in α not within p_1, then in the same manner as above we obtain another polygon p_2 having similar properties, and so on. These polygons can not have an interior point in common, and hence a segment can be a side common to not more than two of them. Hence the process exhausts the segments of $[\sigma]$ and we obtain a finite set of polygons such that every interior point of p in α lies within one of them. A decomposition of these polygons into triangles gives the set of triangles specified in the theorem.

63. *If a plane α contains an interior point of a polyhedron p, then* $[t]_{p,\alpha}$, *together with $[t]$ of p as decomposed by α, form a decomposition of p.*

Proof. Consider those triangles of p which lie on one side of the plane α together with $[t]_{p,\alpha}$ and their interior points. Denote this set by $[t]'$. If a side of a triangle in this set does not lie in α, then by the definition of p every such segment is a side of an even number of triangles.

If a side of a triangle is an intersection of α with a face of p, then it is common to exactly two triangles, one of p and one of $[t]_{p,a}$; and if the side lies in α but not in p, it is also common to two triangles.

It remains to consider the case when an edge of p lies in α. Denote the segment by σ_1. We need to consider two cases: when σ_1 is a side of a triangle in $[t]_{p,a}$, and when it is not. In either case it follows directly from the construction used in the proof of (52) that σ_1 is a side of an even number of triangles in $[t]'$.

Obviously no two triangles of $[t]'$ have an interior point in common, and no one contains an interior point of another. Since the set $[t]'$ is finite, it follows that there must be some subset of it of which no proper subset satisfies the condition that every side of a triangle is common to an even number of triangles. Hence the set $[t]'$ constitutes a set of polyhedrons $[p]'$ such that no two of them have a face in common. Then:

(a) No two polyhedrons of $[p]'$ have an interior point in common. To show this, note first that every polyhedron of $[p]'$ has at least one face in α, and that each triangle of $[t]_{p,}$ is used only once in $[p]'$. From a point Q within a triangle of $[t]_{p,a}$ pass ι half-line which contains no edge or vertex of p and which lies on the same side of α as the interior points of $[p]'$. Then there is a point R on this half-line such that there is no point of p on QR. Then QR lies within exactly one polyhedron of $[p]$. It now follows at once that no point on this ray lies within more than one polyhedron of $[p]'$.

(b) It follows at once also that every interior point of p on the same side of α as R lies within a polyhedron of $[p]'$, and

(c) That every interior point of a polyhedron of $[p]'$ is an interior point of p.

If we now consider that part of p which lies on the opposite side of α from R, we obtain a similar set of $[p]''$ of polyhedrons. Now all interior points of p not in α lie within polyhedrons of the sets $[p]'$ and $[p]''$, and all interior points of p not within these lie on or within the set of triangles $[t]_{p,a}$; that is, they are on the polyhedrons of $[p]'$ and $[p]''$. Hence these two sets form a decomposition of p.

64. PROBLEM. *To decompose any polyhedron into a set of tetrahedrons.*

SOLUTION. If the polyhedron is convex, the problem is solved in (61). If the polyhedron is not convex, the solution is made by indicating how to decompose it into a set of convex polyhedrons.

Let α be a plane determined by a face of the polyhedron p such that not all of the interior points of p lie on the same side of α. Then by (63) $[t]_{p,\alpha}$ decompose p into a set of polyhedrons such that all interior points of each polyhedron lie on the same side of α. If some of the resulting polyhedrons are not convex, we decompose each as before, etc. That this will result in a set of convex polyhedrons follows from the finitude of the number of edges of p.

§ 8.   *Remarks on the Definition of the Polyhedron.*

DEFINITION.   A polyhedron consists of a finite set $[t]$ of triangles, together with their interior points, such that

(1) Every side of a triangle is common to an even number of triangles.

(2) There exists no finite decomposition of the triangles of $[t]$ into other triangles forming a set $[t]'$ such that (1) is true of a proper subset of $[t]'$.

65.   *This definition is not redundant.*

(1) is independent of (2), because (2) is satisfied by a single triangle while (1) is not.

(2) is independent of (1), because (1) is satisfied by the set of triangles forming two distinct tetrahedrons, while (2) is not.

66.   (1) *of the definition is not provable from* (2) *and the following statement:* "*Every side of a triangle is common to at least two triangles.*"

PROOF.   (2) of the definition and the statement "Every side of a triangle is common to at least two triangles" are both satisfied by the following figure, while (1) is not.

Consider an anchor ring with the opening through it closed by that part of a plane which is bounded by the smallest circle whose plane divides the ring into two complete rings. Connect the boundary of a small circle within this circle with the body of the anchor ring by means of a tube which shall contain no interior points of the large circle except the small circle in its interior. If these surfaces are formed by triangles instead of by continuously curved surfaces, we have a set of triangles which satisfy every condition of the definition except that its edges are not all common to an even number of triangles. Obviously this set of triangles does not separate space into two sets.

In terms of the language of this paper a figure corresponding to the above may be described as follows:

Consider a tetrahedron $OABC$. Through A_1, B_1, C_1, points on AO, BO, CO respectively, pass a plane. Let the three non-collinear points A_2, B_2, C_2 within the triangle ABC and the point O determine three triangles. Let the segments OA_2, OB_2, OC_2 meet the interior of the triangle $A_1 B_1 C_1$ in the points

FIG. 11.
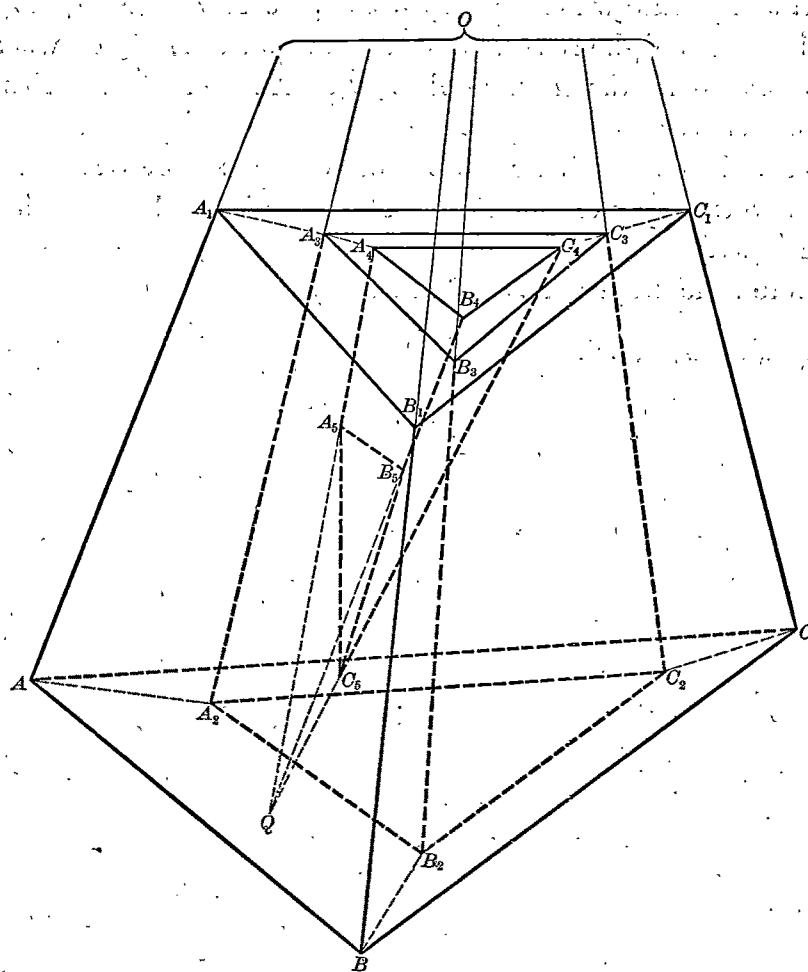
A_3, B_3 and C_3 respectively. Arrange the notation so that the segment $A_1 A_3$ does not meet $A_3 B_3$, $B_3 C_3$ or $C_3 A_3$, and so that $A A_2$ does not intersect $A_2 B_2$, $B_2 C_2$ or $C_2 A_2$. A polyhedron is formed by the triangles into which the following polygons may be decomposed: $ACBB_2C_2A_2A$, AA_2B_2BA, $A_1C_1B_1B_3C_3A_3A_1$, $A_1A_3B_3B_1A_1$, ABB_1A_1A, BCC_1B_1B, CAA_1C_1C, $A_2B_2B_3A_3A_2$, $B_2C_2C_3B_3B_2$ and $C_2A_2A_3C_3C_2$. (This polyhedron corresponds to the anchor ring.)

Call this set of triangles $[t]$.   From Q, any interior point of $A A_2 B_2 B A$, draw segments to the non-collinear points A_4, B_4, C_4 within the triangle $A_3 B_3 C_3$, meeting the interior of the quadrilateral $A_2 B_2 B_3 A_3 A_2$ in the points A_5, B_5, C_5 respectively.   We now adjoin to the set of triangles $[t]$, suitably decomposed, the set of triangles into which we may decompose $A_4 B_4 B_5 A_5 A_4$, $B_4 C_4 C_5 B_5 B_4$ and $C_4 A_4 A_5 C_5 C_4$, and the remainder of the interior of $A_3 B_3 C_3$ when $A_4 B_4 C_4$ is removed, and omit the triangle $A_5 B_5 C_5$.   This set of triangles have the following properties:

(1) Every side of a triangle is common to two or more triangles.   (All are common to two triangles except $A_3 B_3$, $B_3 C_3$, $C_3 A_3$, each of which is common to three triangles.)

(2) There is no proper subset of which (1) is true.

BOSTON, MASS., *January* 11, 1910.

On the Solutions of Certain Types of Linear Differential Equations with Periodic Coefficients.

By F. R. Moulton [*] and W. D. MacMillan.

§ 1. *Introduction.*

Most of the methods which are employed for finding the solutions of differential equations were devised in order to solve the practical problems which arise in celestial mechanics. It is sufficient to mention in this connection the expansion of the solutions as power series in the independent variable, as power series in parameters, the method of the variation of parameters in the general non-linear case, and the method of successive approximations. The first two and the last were used formally by the founders of the analytic theory of the motions of the planets—Clairaut, d'Alembert, and Euler—and the third was given its general formulation by Lagrange at the end of the eighteenth century, and its widest application by Delaunay in his Lunar Theory, in the middle of the nineteenth century. All of these processes were extensively employed in celestial mechanics for obtaining practical results, without any inquiry being made regarding the circumstances and realm of their validity. Indeed, it was as late as 1842 that Cauchy[†] began laying the foundations of the modern theories of differential equations.

The recent contributions to the theory of differential equations, which have been stimulated by problems in celestial mechanics, have come chiefly from the hands of Hill and Poincaré. In 1877 Hill[‡] published privately at Cambridge, Mass., his famous investigation of the motion of the lunar perigee. In this memoir he treated with rare skill the differential equation

$$\frac{d^2w}{dt^2} + \theta(t)w = 0,$$

where θ is a simply periodic function of t. About the same time Hermite* discovered the form of the solution of Lamé's equation, which has a doubly periodic coefficient. Starting from Hermite's results Picard† showed that *in general* a fundamental set of solutions of a linear differential equation of the n-th order having doubly periodic coefficients of the first kind can be expressed in terms of doubly periodic functions of the second kind. In 1883 Floquet‡ published a complete discussion of the character of the solutions of a homogeneous linear differential equation of the n-th order having simply periodic coefficients. In this memoir Floquet gave not only the form of the solution in general, but he considered in detail the forms of the solutions when the fundamental equation has multiple roots. The forms of the solutions being thus known, the efforts of later writers have been directed toward the discovery of practical means for their actual construction. Among those who have discussed the problem of finding the solutions of Hill's equation we may mention Lindemann§, Lindstedt||, Bruns¶, Callandreau**, Stieltjes††, and Harzer‡‡.

In all of these investigations a large amount of attention has been devoted to finding the roots of the fundamental equation, or equivalent transcendentals. Hill determined them from an infinite determinant which he first introduced into analysis in this connection; Lindstedt found them from an infinite continued fraction§§. In all cases these transcendentals were computed first, and then the solutions were found later. It should be noted also that the processes are valid only under certain special conditions which, fortunately, are satisfied in the case of Hill's equation. The problem is treated in a much more general way in Poincaré's *Les Méthodes Nouvelles de la Mécanique Céleste*, Vol. I, Chapter II, Sec. 29, and Chapter IV.

In dynamical problems involving accelerations simultaneous differential equations of the second order naturally arise, but it is easy to reduce them to twice the number of simultaneous equations of the first order. Now n simul-

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* *Comptes Rendus*, 1877 et seq.

† *Comtes Rendus*, 1879–1880; *Journal für Mathematik*, Vol. XC (1881).

‡ *Annales de l'École Normale Supérieure*, 1883–1884.

§ *Mathematische Annalen*, Vol. XXII (1883), p. 117.

|| *Astronomische Nachrichten*, No. 2503 (1883), and *Mémoires de l'Académie de St. Pétersbourg*, Vol. XXI, No. 4.

¶ *Astronomische Nachrichten*, Nos. 2533 and 2553 (1883).

** *Ibid.*, No. 2547 (1883).

†† *Ibid.*, Nos. 2601 and 2609 (1884).

‡‡ *Ibid.*, Nos. 2850 and 2851 (1888).

§§ See Tisserand's *Mécanique Céleste*, Vol. III, Chapter I.

taneous differential equations of the first order include one differential equation of the n-th order, for the latter can always be reduced to the former. Hence we shall treat here as the general case and the one most simply connecting with dynamical problems a set of simultaneous linear differential equations of the first order having simply periodic coefficients. We shall find the character of the solutions of the differential equations without further restrictions by a very direct process. Then, simple and convenient methods are given for constructing the solutions in all cases in which the coefficients of the differential equations are expansible as power series in a parameter μ, and the terms not depending upon μ (at least in a large part of the discussion) are constants with respect to the independent variable. The linear differential equations having periodic coefficients which arise in celestial mechanics,[*] of which Hill's equation is a simple example, belong to this class.

§.2. *The Fundamental Equation.*

We shall consider the differential equations

$$x_i' = \sum_{j=1}^{n} \theta_{ij}(t)x_j, \qquad i = 1, \ldots, n, \tag{1}$$

where x_i' is the derivative of x_i with respect to the independent variable t, and where the θ_{ij} are uniform analytic functions of t and are periodic with the period 2π. Let

$$x_{i1} = \phi_{i1}(t), \quad \ldots, \quad x_{in} = \phi_{in}(t), \qquad i = 1, \ldots, n,$$

be a fundamental set of solutions of (1), where $x_{ij} = \phi_{ij}(t)$, $i = 1, \ldots, n$, is the j-th solution. The determinant of the fundamental set,

$$\Delta = |\phi_{ij}|,$$

is found, by taking the derivative and reducing by means of (1), to satisfy the relation

$$\Delta' = \Delta \sum_{i=1}^{n} \theta_{ii}(t);$$

whence [†]

$$\Delta = \Delta_0 \, e^{\int_{t_0}^{t} \sum_{i=1}^{n} \theta_{ii} \, dt}. \tag{2}$$

Hence Δ can become zero or infinite only at a singularity of some $\theta_{ii}(t)$.

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[*] Some of the methods exhibited here were devised in connection with problems raised in the theory of periodic orbits. They were first applied to Hill's equation by Moulton in a paper whose abstract is in *Bull. of the Am. Math. Soc.*, Vol. XIII (1906-7), p. 71, and were later extended by our joint investigations to the general case.

[†] Darboux, *Comptes Rendus*, Vol. XC (1880), p. 526.

We now start from a particular set of solutions satisfying the initial con-, ditions

$$\phi_{ii}(0) = 1, \quad \phi_{ij}(0) = 0, \qquad \text{if } j \neq i. \tag{3}$$

It is clear that a set of n solutions satisfying these initial conditions can be constructed, and since for them $\Delta_0 = 1$ is distinct from zero, they form a *fundamental set of solutions.*

Let us make the transformation

$$x_i = e^{\alpha t} y_i, \tag{4}$$

α being an undetermined constant. Then equations (1) become

$$y_i' + \alpha y_i = \sum_{j=1}^{n} \theta_{ij} y_j. \tag{5}$$

Any solution of (5) can be written in the form

$$y_i = e^{-\alpha t} \sum_{j=1}^{n} A_j \phi_{ij}(t), \qquad i = 1, \ldots, n, \tag{6}$$

where the A_j are suitably chosen constants.

We now inquire whether it is possible to determine α and the A_j so that the y_i, as defined by (6), shall be periodic with the period 2π. From the form of (5) it is clear that sufficient conditions for the periodicity of the y_i with the period 2π are

$$y_i(2\pi) - y_i(0) = 0, \qquad i = 1, \ldots, n. \tag{7}$$

Imposing these conditions on (6), we get

$$\sum_{j=1}^{n} A_j [\phi_{ij}(2\pi) - e^{2\alpha\pi} \phi_{ij}(0)] = 0, \qquad i = 1, \ldots, n. \tag{8}$$

In order that these equations may have a solution other than $A_1 = \ldots = A_n = 0$, the determinant of the coefficients of the A_j must equal zero. Making use of (3), representing $\phi_{ij}(2\pi)$ simply by ϕ_{ij}, and letting $e^{2\alpha\pi} = s$, the determinant is

$$D = \begin{vmatrix} \phi_{11} - s, & \phi_{12} & , & \cdots, & \phi_{1n} \\ \phi_{21} & , & \phi_{22} - s, & \cdots, & \phi_{2n} \\ \cdots, & \cdots, & \cdots, & \cdots \\ \phi_{n1} & , & \phi_{n2} & , & \cdots, & \phi_{nn} - s \end{vmatrix} = 0. \tag{9}$$

This is the *fundamental equation* associated with the period 2π and is of degree n in s. It admits neither $s = 0$ nor $s = \infty$ as a root, since the absolute term is the value of the determinant of the fundamental set of solutions at the regular point $t = 2\pi$, and the coefficient of s^n is $(-1)^n$.

§3. *Form of the Solutions.*

Suppose the roots of (9) are $s_1, \ldots, s_n$ and that they are all distinct. For each of them there is therefore at least one first minor of D which is distinct from zero, and hence for each of them the ratios of the A_j can be determined from (8) so that the y_i shall be periodic. In this way n solutions are obtained which can be shown to constitute a fundamental set.

Suppose now that $s_2 = s_1$ and that all the other roots of (9) are distinct. There are two cases to be considered, according as all of the first minors of D vanish or do not vanish for $s = s_1$. Suppose all the first minors of D vanish for $s = s_1$; since by hypothesis $s = s_1$ is a double root and not a triple root, there is at least one second minor of D which does not vanish for $s = s_1$. Therefore two of the A_j can be taken arbitrarily and the remaining $n-2$ can be expressed in terms of them so that the y_i shall be periodic. We thus obtain two distinct solutions of the form

$$x_{i1} = e^{\alpha_1 t} y_{i1} \quad \text{and} \quad x_{i2} = e^{\alpha_1 t} y_{i2}, \qquad i = 1, \ldots, n, \qquad (10)$$

where the y_{i1} and y_{i2} are periodic.

Suppose $s = s_1$ is a double root of (9) and that not all its first minors vanish for $s = s_1$. Then there is but one solution of the form

$$x_{i1} = e^{\alpha_1 t} y_{i1}.$$

The y_{i1} are expressible linearly in terms of the ϕ_{ij} by (6). Let the notation be chosen so that the minor which is not zero is formed from the elements of the last $n-1$ columns. Then A_1 must be distinct from zero in order to avoid the trivial case in which all the A_j are zero.

Then we take as a new set of solutions

$$x_{i1} = e^{\alpha_1 t} y_{i1}, \quad x_{ij} = \phi_{ij}(t), \qquad i = 1, \ldots, n; \, j = 2, \ldots, n. \qquad (11)$$

These solutions constitute a fundamental set, for their determinant is

$$e^{\alpha_1 t} \begin{vmatrix} y_{11}, & \phi_{12}, & \ldots, & \phi_{1n} \\ y_{21}, & \phi_{22}, & \ldots, & \phi_{2n} \\ \ldots, & \ldots, & \ldots, & \ldots \\ y_{n1}, & \phi_{n2}, & \ldots, & \phi_{nn} \end{vmatrix},$$

which becomes, by means of (6), $A_1 |\phi_{ij}|$, which is distinct from zero. Starting with this fundamental set, the fundamental equation is

$$D = (s - s_1)\, D_1 = (s - s_1)\, \begin{vmatrix} y_{11}(0)\,, & \phi_{12} & , & \cdots , & \phi_{1n} \\ y_{21}(0)\,, & \phi_{22} - s\,, & & \cdots , & \phi_{2n} \\ \cdots\, , & \cdots\cdots , & & \cdots , & \cdots \\ y_{n1}(0), & \phi_{n2} & , & \cdots , & \phi_{nn} - s \end{vmatrix} = 0. \qquad (12)$$

Since $s = s_1$ is a double root of D, the determinant D_1 has a single factor $s - s_1$. Since (11) constitute a fundamental set, any solution can be expressed in the form

$$x_i = B_1 e^{\alpha_1 t} y_{i1} + \sum_{j=2}^{n} B_j \phi_{ij}, \qquad i = 1, \cdots , n. \qquad (13)$$

Now we make the transformation, corresponding to (4), to get a second solution associated with α_1,

$$x_{i2} = e^{\alpha_1 t}(y_{i2} + t y_{i1}). \qquad (14)$$

Imposing the condition that the x_{i2} shall satisfy (1) we find, since $e^{\alpha_1 t} y_{i1}$ is a solution,

$$y_{i2}' + \alpha_1 y_{i2} = \sum_{j=1}^{n} \theta_{ij} y_{j2} - y_{i1}, \qquad i = 1, \cdots , n.$$

Therefore sufficient conditions that the y_{i2} shall be periodic with the period 2π are

$$y_{i2}(2\pi) - y_{i2}(0) = 0 = -2\pi y_{i1}(0) + \sum_{j=2}^{n} B_j [e^{-2\alpha_1 \pi} \phi_{ij}(2\pi) - \phi_{ij}(0)].$$

Substituting s_1 for $e^{2\alpha_1 \pi}$, we have

$$-2\pi s_1 y_{i1}(0) + \sum_{j=2}^{n} B_j [\phi_{ij}(2\pi) - s_1 \phi_{ij}(0)] = 0, \qquad i = 1, \cdots , n. \qquad (15)$$

The condition that these equations shall be consistent is

$$D_1 = \begin{vmatrix} y_{11}(0)\,, & \phi_{12} & , & \cdots , & \phi_{1n} \\ y_{21}(0)\,, & \phi_{22} - s_1\,, & & \cdots , & \phi_{2n} \\ \cdots\cdots , & \cdots\cdots , & & \cdots , & \cdots \\ y_{n1}(0), & \phi_{n2} & , & \cdots , & \phi_{nn} - s_1 \end{vmatrix} = 0.$$

It was shown in (12) that this determinant has a single root $s = s_1$. Hence not all its first minors are zero. By hypothesis not all minors of the first order formed from the last $n - 1$ columns of D, which are the same as the last $n - 1$ columns of D_1, are zero. Therefore we can solve equations (15) for $B_2, \cdots , B_n$ in terms of $y_{i1}(0)$ which carry an arbitrary constant as a factor. Consequently in this case a second solution depending upon α_1 exists and is expressible in the form (14), where y_{i1} and y_{i2} are both periodic.

When $s = s_1$ is a triple root of $D = 0$, an analogous discussion shows that the three associated solutions are of the form

$$x_{i1} = e^{a_1 t} y_{i1}, \quad x_{i2} = e^{a_1 t} y_{i2}, \qquad\qquad x_{i3} = e^{a_1 t} y_{i3}, \qquad\qquad i = 1, \ldots, n,$$

or

$$x_{i1} = e^{a_1 t} y_{i1}, \quad x_{i2} = e^{a_1 t} y_{i2}, \qquad\qquad x_{i3} = e^{a_1 t} [y_{i3} - t(y_{i1} + y_{i2})], \quad i = 1, \ldots, n,$$

or

$$x_{i1} = e^{a_1 t} y_{i1}, \quad x_{i2} = e^{a_1 t} [y_{i2} + t y_{i1}], \quad x_{i3} = e^{a_1 t} [y_{i3} + t y_{i2} + \tfrac{1}{2} t^2 y_{i1}], \quad i = 1, \ldots, n,$$

according as all the minors of D of the first and second orders vanish for $s = s_1$, or all those of the first order but not all those of the second order vanish, or not all the minors of the first order vanish.

In case $s = s_1$ is a root of multiplicity ν, the associated group of solutions have the form

$$x_{i1} = e^{a_1 t} y_{i1},$$
$$x_{i2} = e^{a_1 t} [y_{i2} + t y_{i1}],$$
$$x_{i3} = e^{a_1 t} [y_{i3} + t y_{i2} + \tfrac{1}{2} t^2 y_{i3}], \tag{16}$$
$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots,$$
$$x_{i\nu} = e^{a_1 t} \left[[y_{i\nu} + t y_{i(\nu-1)} + \cdots + \frac{1}{(\nu-1)!} t^{\nu-1} y_{i1}] \right],$$

where the y_{ij} are periodic with the period 2π, provided not all the minors of D of the first order vanish for $s = s_1$. In case all the minors of D of order $k - 1$, but not all of order k, vanish for $s = s_1$, there are k solutions of the type of the first of (16), the others involving products of t to powers not exceeding $\nu - k$ and the y_{ij}, $i = 1, \ldots, n$, $j = 1, \ldots, \nu$. The details for a single differential equation of order n were given by Flcquet (*loc. cit.*), and the results are similar here.

§ 4.   *The Characteristic Equation when the Coefficients, θ_{ij}, are Power Series in a Parameter μ.*

We shall now assume that the θ_{ij} are expansible as power series in a parameter μ whose coefficients are separately periodic with the period 2π, and that the series converge for all real finite values of t if $|\mu| < \rho$. We shall assume further that for $\mu = 0$, $\theta_{ij} = a_{ij}$, where the a_{ij} are constants. Under these conditions, which are often realized in practice, particularly in celestial mechanics,[*]

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[*] Hill's differential equation, considered first in his paper on the motion of the lunar perigee, is an equation of this type.

the discussion of the character of the solutions can be made so as to lead to a convenient method for their explicit construction.

Consider the equations

$$x_i' = \sum_{j=1}^{n} \theta_{ij} x_j = \sum_{j=1}^{n} [a_{ij} + \sum_{k=1}^{\infty} \theta_{ij}^{(k)} \mu^k] x_j, \qquad i = 1, \ldots, n, \qquad (17)$$

where the a_{ij} are constants. When $\mu = 0$ they have the solution

$$x_i^{(0)} = c_i e^{a^{(0)} t},$$

where $a^{(0)}$ is any one of the roots of the characteristic equation

$$\begin{vmatrix} a_{11} - a^{(0)}, & a_{12} & , & \ldots, & a_{1n} \\ a_{21} & , & a_{22} - a^{(0)}, & \ldots, & a_{2n} \\ \ldots\ldots, & \ldots\ldots, & \ldots, & \ldots \\ a_{n1} & , & a_{n2} & , & \ldots, & a_{nn} - a^{(0)} \end{vmatrix} = 0. \qquad (18)$$

If the n roots of this equation are distinct, a fundamental set of solutions of (17) for $\mu = 0$ is

$$x_{ij}^{(0)} = c_{ij} e^{a_j^{(0)} t}, \qquad i, j = 1, \ldots, n,$$

where the c_{ij} can be taken so that $|c_{ij}| = 1$.

By Poincaré's extension of Cauchy's theorem,* equations (17) can be integrated as power series in μ which will converge, for $0 \lesseqgtr t \leq T$, T any arbitrary finite value, provided $|\mu| < R$, R depending upon T. Hence we can write the general solution of (17) in the form

$$x_i = \sum_{j=1}^{n} A_j [c_{ij} e^{a_j^{(0)} t} + \sum_{k=1}^{\infty} x_{ij}^{(k)} \mu^k], \qquad (19)$$

where the A_j are the constants of integration, and the $x_{ij}^{(k)}$ are functions of t depending on the θ_{ij}. We can take the initial conditions so that

$$x_{ij}(0) = \sum_{k=0}^{\infty} x_{ij}^{(k)}(0) \mu^k \equiv c_{ij}.$$

Therefore

$$x_{ij}^{(0)}(0) = c_{ij}, \quad x_{ij}^{(k)}(0) = 0, \qquad k = 1, \ldots, \infty.$$

Now we make the transformation

$$x_i = e^{at} y_i,$$

after which the differential equations and their solutions become

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* *Les Méthodes Nouvelles de la Mécanique Céleste*, Vol. I, Chapter II.

$$y_i' + \alpha y_i = \sum_{j=1}^{n} [a_{ij} + \sum_{k=1}^{\infty} \theta_{ij}^{(k)} \mu^k] y_i, \qquad i = 1, \ldots, n;$$

$$y_i = \sum_{j=1}^{n} A_j e^{-\alpha t} [c_{ij} e^{\alpha^{(0)} t} + \sum_{k=1}^{\infty} x_{ij}^{(k)}(t) \mu^k], \qquad i = 1, \ldots, n. \qquad (20)$$

Imposing the conditions that the y_i shall be periodic with the period 2π, viz., $y_i(2\pi) - y_i(0) = 0$, we get

$$0 = \sum_{j=1}^{n} A_j [c_{ij}(e^{2(\alpha_j^{(0)} - \alpha)\pi} - 1) + e^{-2\alpha\pi} \sum_{k=1}^{\infty} x_{ij}^{(k)}(2\pi) \mu^k], \qquad i = 1, \ldots, n. \qquad (21)$$

In order not to have the trivial case where all the A_j are zero, we must set the determinant

$$D = | [c_{ij}(e^{2(\alpha_j^{(0)} - \alpha)\pi} - 1) + e^{-2\alpha\pi} \sum_{k=1}^{\infty} x_{ij}^{(k)}(2\pi) \mu^k] | = 0, \qquad (22)$$

which is a condition upon the undetermined constant α. This equation has an infinite number of solutions; for if it is satisfied by $\alpha = \alpha_0$, it is also satisfied by $\alpha = \alpha_0 + \nu \pi \sqrt{-1}$, where ν is any integer. All distinct solutions can be obtained with ν equal to zero; the others amount simply to taking periodic factors from the y_i. The fundamental equation corresponding to (22) is obtained by putting $e^{2\alpha\pi} = s$. If the n values of s satisfying the fundamental equation are distinct, the corresponding values of α are distinct, but not necessarily the converse. We shall use only those n values of α which reduce to the $\alpha_j^{(0)}$ for $\mu = 0$, the n $\alpha_j^{(0)}$ being uniquely determined by (18).

In the event that two of the roots of the characteristic equation, say $\alpha_1^{(0)}$ and $\alpha_2^{(0)}$, are equal, the solutions are either of the form (19), where $\alpha_2^{(0)} = \alpha_1^{(0)}$, or

$$x_i = A_1 [c_{i1} e^{\alpha_1^{(0)} t} + \sum_{k=1}^{\infty} x_{i1} \mu^k] + A_2 [(c_{i2} + t c_{i1}) e^{\alpha_1^{(0)} t} + \sum_{k=1}^{\infty} x_{i2}^{(k)} \mu^k]$$

$$+ \sum_{j=3}^{n} A_j [c_{ij} e^{\alpha_j^{(0)} t} + \sum_{k=1}^{\infty} x_{ij}^{(k)} \mu^k]. \qquad (23)$$

After making the transformation $x_i = e^{\alpha t} y_i$, writing out the solutions for y_i from (23), and imposing the periodicity conditions, $y_i(2\pi) - y_i(0) = 0$, on the y_i, we find that if not all the A_j are zero either (22) is satisfied or

$$D = | [c_{i1}(e^{2(\alpha_1^{(0)} - \alpha)\pi} - 1) + e^{-2\alpha\pi} \sum_{k=1}^{\infty} x_{i1}^{(k)}(2\pi) \mu^k],$$

$$[c_{i2}(e^{2(\alpha_1^{(0)} - \alpha)\pi} - 1) + 2\pi c_{i1} e^{2(\alpha_1^{(0)} - \alpha)\pi} + e^{-2\alpha\pi} \sum_{k=1}^{\infty} x_{i2}^{(k)}(2\pi) \mu^k], \ldots | = 0, \qquad (24)$$

according as the solutions for $\mu = 0$ are of the form (19) or (23), where the terms of the determinant not written are of the same form as those in (22). When for $\mu = 0$ the characteristic equation has a root of higher order of multiplicity, a corresponding discussion must be made. If for $\mu = 0$ the characteristic equation has several multiple roots, a corresponding discussion must be made for each associated group of solutions.

§ 5. *Solutions when $\alpha_i^{(0)}$ Are Distinct and $\alpha_i^{(0)} - \alpha_j^{(0)} \not\equiv 0 \mod \sqrt{-1}$.*

The part of (22) independent of μ is

$$D_0 = |c_{ij}(e^{2(\alpha_j^{(0)} - \alpha)\pi} - 1)| = |c_{ij}| \prod_{j=1}^{n}(e^{2(\alpha_j^{(0)} - \alpha)\pi} - 1),$$

and the initial conditions have been taken so that $|c_{ij}| = 1$. If (22) were an identity in μ, its n solutions would be the n solutions of $D_0 = 0$, viz., $\alpha = \alpha_j^{(0)}$. In the general case in which it is not an identity in μ let

$$\alpha = \alpha_k^{(0)} + \beta_k. \tag{25}$$

Then we get

$$D = D_0 + \mu F_k(\beta_k, \mu) = (e^{-2\beta_k \pi} - 1)\prod_{j=1}^{n}{}'(e^{2(\alpha_j^{(0)} - \alpha_k^{(0)} - \beta_k)\pi} - 1)$$
$$+ \mu F_k(\beta_k, \mu) = 0, \qquad j \neq k, \tag{26}$$

where $F_k(\beta_k, \mu)$ is a power series in μ and β_k, converging for

$$|\beta_k| < \infty, \qquad |\mu| < \rho > 0.$$

Since by hypothesis no $\alpha_j^{(0)} - \alpha_k^{(0)}$ equals an imaginary integer, the expansion of (26) as a power series in β_k and μ contains a term in β_k of the first degree and no term independent of both β_k and μ. Therefore by the theory of implicit functions (26) can be solved uniquely for β_k as a power series in μ of the form

$$\beta_k = \mu P_k(\mu), \tag{27}$$

which converges for $|\mu| > 0$ but sufficiently small. When we substitute this value of $\alpha = \alpha_k^{(0)} + \beta_k$ back in (21), we have n homogeneous linear equations among the A_j whose determinant is zero, but whose first minors, for $\mu = 0$, are not all zero; since, by hypothesis, for $\mu = 0$ the roots of the determinant set equal to zero are all distinct and no two differ by an imaginary integer. Therefore the ratios of the A_j are uniquely determined by these equations as power series in μ, converging for $|\mu|$ sufficiently small. Substituting the ratios

of the A_j in (20), we have the particular solution y_{ik}, $i = 1, \ldots, n$, carrying one arbitrary constant and expanded as a power series in μ. Hence we may write it

$$y_{ik} = a_k \sum_{j=0}^{\infty} y_{ik}^{(j)} \mu^j. \tag{28}$$

Since the periodicity conditions have been satisfied,

$$y_{ik}(t + 2\pi) - y_{ik}(t) = \sum_{j=0}^{\infty} [y_{ik}^{(j)}(t + 2\pi) - y_{ik}^{(j)}(t)] \mu^j = 0$$

for every μ whose modulus is sufficiently small and for all t. Therefore

$$y_{ik}^{(j)}(t + 2\pi) - y_{ik}^{(j)}(t) = 0, \qquad j = 0, \ldots, \infty,$$

from which it follows that each $y_{ik}^{(j)}$, $j = 0, \ldots, \infty$, is separately periodic. A solution is found similarly for each $a_j^{(0)}$.

§ 6. *Solutions when no Two $a_j^{(0)}$ Are Equal but when $a_2^{(0)} - a_1^{(0)} \equiv 0$ mod $\sqrt{-1}$*

Suppose two roots of the characteristic equation (18) for $\mu = 0$, say $a_1^{(0)}$ and $a_2^{(0)}$, differ by an imaginary integer and that there is no other such congruence among the $a_j^{(0)}$. Then the equation corresponding to (26) becomes

$$D = (e^{-2\beta_1 \pi} - 1)^2 \prod_{j=3}^{n} (e^{2(a_j^{(0)} - a_1^{(0)} - \beta_1)\pi} - 1) + \beta_1 \mu F_1(\beta_1, \mu) + \mu^2 F_2(\beta_1, \mu) = 0. \tag{29}$$

The term of lowest degree in β_1 alone is $+4\pi^2 \beta_1^2$. The terms independent of β carry μ^2 as a factor, for every element of the first two columns of D, equation (24), in this case carries either β_1 or μ as a factor. In order to get the terms in μ alone we suppress those involving β_1, after which we get a factor μ from each of the first two columns. In general the term of lowest degree in μ alone will be in this case of the second degree.

In a similar manner if p of the $a_j^{(0)}$ are congruent to zero mod $\sqrt{-1}$, then the term of lowest degree in β_1 alone is of degree p, and in μ alone it is at least of degree p.

The problem of the form of the solution of (29) is one of implicit functions. Writing the first terms explicitly, we have

$$\beta_1^2 + \gamma_{11} \beta_1 \mu + \gamma_{02} \mu^2 + \text{terms of higher degree} = 0,$$

where γ_{11}, γ_{02}, $\ldots$ are constants independent of β_1 and μ. The quadratic terms may be factored and we get

$$(\beta_1 - b_1 \mu)(\beta_1 - b_2 \mu) + \text{terms of higher degree} = 0.$$

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If b_1 and b_2 are distinct, the two solutions have the form

$$\beta_{11} = b_1\mu + \mu^2 P_1(\mu), \;\Big\rbrace$$
$$\beta_{12} = b_2\mu + \mu^2 P_2(\mu), \;\Big\rbrace \qquad\qquad (30)$$

where P_1 and P_2 are power series in μ which converge if $|\mu|$ is sufficiently small. In this case the solutions are found precisely as in § 5. '

If $b_1 = b_2$, the character of the solution depends upon terms of higher degree. It will proceed in powers of $\pm \sqrt{\mu}$ unless the $\theta_{ij}^{(1)}$ satisfy special conditions. But for special values of the $\theta_{ij}^{(1)}$ it may proceed according to integral powers of μ. We shall consider in detail the case where the series is in powers of $\pm\sqrt{\mu}$.

We see that the expansion of α_1 as a power series in $\sqrt{\mu}$ will contain no term in $\sqrt{\mu}$ to the first power, but will have the form

$$\alpha_1 = \alpha_1^{(0)} + 0\mu^{\frac{1}{2}} + \alpha_1^{(1)}\mu + \alpha_1^{(3)}\mu^{\frac{3}{2}} + \ldots \ldots.$$

Suppose this series has been obtained from equation (29). Then since not all of the first minors of D are zero the ratios of the A_j will be determined from (21). Suppose $\mu\delta$ is a non-vanishing first minor of D formed from the elements of its last $n-1$ columns. Then it follows from the form of (22) that, solving (21), we get

$$A_2 = \frac{\mu\delta_2}{\mu\delta} A_1, \quad A_j = \frac{\mu^2\delta_j}{\mu\delta} A_1, \qquad j = 3, \ldots, n,$$

where

$$\delta = \delta^{(0)} + \delta^{(3)}\mu^{\frac{1}{2}} + \delta^{(1)}\mu + \ldots,$$
$$\delta_j = \delta_j^{(0)} + \delta_j^{(3)}\mu^{\frac{1}{2}} + \delta_j^{(1)}\mu + \ldots, \qquad j = 2, \ldots, n.$$

Substituting these series for the A_j in (20), we find that the y_{i1} are developable as series of the form

$$y_{i1} = y_{i1}^{(0)} + y_{i1}^{(3)}\mu^{\frac{1}{2}} + y_{i1}^{(1)}\mu + \ldots, \qquad i = 1, \ldots, n.$$

Therefore the y_{i1} carry terms in $\mu^{\frac{1}{2}}$, although the term in $\sqrt{\mu}$ is absent in the expression for α_1.

If all the first minors obtained from D when the first column is suppressed are zero, and if there is a first minor which is distinct from zero when the second column is suppressed, the results are precisely the same. But suppose all the first minors obtained by omitting in turn the first and second columns are zero. Since the determinant has simple roots, there is at least one minor of the first order which is distinct from zero. Suppose it is obtained when the kth column

is suppressed. It follows from (22) that when $\alpha = \alpha_1$ it will carry the factor μ^2; let it be $\mu^2\delta$. Then solving (21), we get

$$A_1 = \frac{\mu\delta_1}{\mu^2\delta} A_k, \quad A_2 = \frac{\mu\delta_2}{\mu^2\delta} A_k, \quad A_j = \frac{\mu^2\delta_j}{\mu^2\delta} A_k, \qquad j = 3, \ldots, n,$$

where δ_1, δ_2, δ_j do not in general vanish for $\mu = 0$. It follows from the first two equations that A_k must carry μ as a factor. Hence in this case the y_{i1} have the same form as before. The y_{i2} have the same properties. These properties of the y_{i1} and the y_{i2} are necessary for the construction of the solution.

We now return to the consideration of (29). If the discriminant of the quadratic terms of (29),

$$D = \gamma_{11}^2 - 4\gamma_{20},$$

is distinct from zero, the solutions are in integral powers of μ, and at least one of them starts with a term of the first degree in μ. But if the discriminant is zero, then the character of the solutions depends upon the coefficients of terms of higher degree. They may be either in integral powers of μ or in powers of $\sqrt{\mu}$. If the solutions are in $\sqrt{\mu}$, they are real when μ has one sign and complex when it has the other. But if the solutions are in integral powers of μ, they are either real or complex for both positive and negative values of μ.

In all cases we get two solutions associated with the root $\alpha_1^{(0)}$. We should also get two solutions if we started from the root $\alpha_2^{(0)}$, but it follows from the form of (22) that they would not differ from those obtained by starting from $\alpha_1^{(0)}$. For each $\alpha_1^{(0)} + \beta_1$ the ratios of the A_j are determined from (21), and the results substituted in (20) give the y_i. The solutions associated with $\alpha_3^{(0)}, \ldots, \alpha_n^{(0)}$ are found as in the preceding case. If there are several groups of $\alpha^{(0)}$ in which these congruences exist, the discussion is made similarly for each one of them.

§ 7. *Solutions when $\alpha_1^{(0)}$ is a Multiple Root.*

Suppose that two and only two of the $\alpha_j^{(0)}$, viz. $\alpha_1^{(0)}$ and $\alpha_2^{(0)}$, are equal, and that there are none of the congruences treated in § 6. Then for $\mu = 0$ we get from (22) and (24) either

$$D_0 = |\, c_{i1}(e^{2(\alpha_1^{(0)}-\alpha)\pi} - 1), \quad c_{i2}(e^{2(\alpha_1^{(0)}-\alpha)\pi} - 1), \quad \ldots, \quad c_{ij}(e^{2(\alpha_j^{(0)}-\alpha)\pi} - 1), \quad \ldots \,|,$$

or

$$D_0 = |\, c_{i1}(e^{2(\alpha_1^{(0)}-\alpha)\pi} - 1), \quad c_{i2}(e^{2(\alpha_1^{(0)}-\alpha)\pi} - 1) + 2\pi c_{i1}e^{2(\alpha_1^{(0)}-\alpha)\pi}, \quad \ldots,$$
$$c_{ij}(e^{2(\alpha_j^{(0)}-\alpha)\pi} - 1), \quad \ldots \,|,$$

both of which, by the theory of determinants, reduce to

$$D_0 = |c_{ij}| (e^{2(\alpha_1^{(0)}-\alpha)\pi} - 1)^2 \prod_{j=3}^{n} (e^{2(\alpha_j^{(0)}-\alpha)\pi} - 1), \quad (|c_{ij}| = 1). \tag{31}$$

If we let $\alpha = \alpha_1^{(0)} + \beta_1$, as before, and expand as a power series in β_1, we find that the term of lowest degree in β_1 alone is $4\pi^2\beta_1^2$. When the determinant D is of the form (22) with $\alpha_2^{(0)} = \alpha_1^{(0)}$, the term of lowest degree in μ alone is at least of the second degree; but when D is of the form (24), which is the general case, the term of lowest degree in μ alone is in general of the first degree. Except in the special cases, the solutions for β_1 are therefore of the form

$$\left. \begin{aligned} \beta_{11} &= \mu^{\frac{1}{2}} P(\mu^{\frac{1}{2}}), \\ \beta_{12} &= -\mu^{\frac{1}{2}} P(-\mu^{\frac{1}{2}}), \end{aligned} \right\} \tag{32}$$

where P is a power series in $\mu^{\frac{1}{2}}$ containing a term independent of μ. The treatment of the special cases proceeds as in §6. Substituting these expansions for $\alpha_1 = \alpha_1^{(0)} + \beta_1$ in (21) the ratios of the A_j are determined as power series in $\sqrt{\mu}$, and these results substituted in (21) give the y_{i1} and y_{i2} as power series in $\sqrt{\mu}$.

If, for $\mu = 0$, p roots, $\alpha_1^{(0)}, \ldots, \alpha_p^{(0)}$, are equal, then for these roots the expansion of D starts with β_1^p as the term of lowest degree in β_1 alone, and, except in special cases corresponding to those above mentioned when two roots are equal, the term of lowest degree in μ alone is of the first degree. Consequently in general for $\alpha_1^{(0)} = \ldots = \alpha_p^{(0)}$ we have

$$\beta_{1j} = \varepsilon^j \mu^{\frac{1}{p}} P(\varepsilon^j \mu^{\frac{1}{p}}), \qquad j = 0, \ldots, p-1,$$

where ε is a pth root of unity.

§8. *Solutions when there Are Equalities and Congruences among the Roots of the Characteristic Equation.*

Suppose, for example, that $\alpha_1^{(0)} = \alpha_2^{(0)}$ and $\alpha_3^{(0)}$ differs from $\alpha_1^{(0)}$ and $\alpha_2^{(0)}$ by an imaginary integer, and that there are no other equalities or congruences among the $\alpha_j^{(0)}$. There are two cases, (a) where the solutions are of the form (19) with $\alpha_1^{(0)} = \alpha_2^{(0)}$, and (b) where the solutions are of the form (23).

Case (a). In this case $D_0 = 0$ becomes

$$D_0 = | c_{i1}(e^{2(\alpha_1^{(0)}-\alpha)\pi} - 1), \quad c_{i2}(e^{2(\alpha_1^{(0)}-\alpha)\pi} - 1), \quad c_{i3}(e^{2(\alpha_3^{(0)}-\alpha)\pi} - 1), \ldots,$$
$$c_{ij}(e^{2(\alpha_j^{(0)}-\alpha)\pi} - 1), \ldots | = 0.$$

Putting $\alpha = \alpha_1^{(0)} + \beta_1$, we find that the term of lowest degree in β_1 is $- 8\pi^3\beta_1^3$. In order to get the term of D of lowest degree in μ we put $\beta_1 = 0$. Then the first three columns of D are divisible by μ, while the others do not contain μ as factor. Consequently the term of lowest degree in μ is of the third degree at least. Moreover, since the first three columns of D vanish with $\beta_1 = \mu = 0$, there are no terms lower than the third degree in β_1 and μ. Hence in general D, in case (a), is of the form

$$D = \beta_1^3 + \gamma_{21}\beta_1^2\mu + \gamma_{12}\beta_1\mu^2 + \gamma_{03}\mu^3 + \ldots = 0. \qquad (33)$$

Since the coefficients of this equation are real, it always has at least one real solution for β_1, vanishing with μ. The details of the various special cases are simply those of implicit functions. In general the three values of β_1 are expansible in integral powers of μ.

Case (b). In this case $D_0 = 0$ becomes

$$D_0 = |\, c_{i1}(e^{2(\alpha_1^{(0)}-\alpha)\pi} - 1), \quad c_{i2}(e^{2(\alpha_1^{(0)}-\alpha)\pi} - 1) + 2\pi c_{i1}e^{2(\alpha_1^{(0)}-\alpha)\pi}, \quad c_{i3}(e^{2(\alpha_3^{(0)}-\alpha)\pi} - 1), \ldots \,| = 0.$$

By the theory of determinants this equation reduces to

$$D_0 = (e^{2(\alpha_1^{(0)}-\alpha)\pi} - 1)^2(e^{2(\alpha_3^{(0)}-\alpha)\pi} - 1)\prod_{j=4}^{n}(e^{2(\alpha_j^{(0)}-\alpha)} - 1) = 0.$$

Introducing β_1 as before, we find that the term of lowest degree in β_1 alone is $-8\pi^3\beta_1^3$. But when the terms involving μ are retained in D, the terms $2\pi c_{i1}e^{2(\alpha_1^{(0)}-\alpha)\pi}$ can not be eliminated from the second column. Hence only the first two columns vanish for $\beta_1 = \mu = 0$, and therefore in general in this case the expansion of D will contain a term in μ^2 alone. Since the first two columns vanish for $\beta_1 = \mu = 0$, there will be no terms of degree lower than the second in β_1 and μ. Hence in general D, in case (b), has the form

$$D = \beta_1^3 + \gamma_{11}\beta_1\mu + \gamma_{02}\mu^2 + \ldots = 0. \qquad (34)$$

In the general case in which $\gamma_{11} \neq 0$ and $\gamma_{02} \neq 0$ there is one real solution in integral powers of μ and two solutions in $\sqrt{\mu}$. The two latter are real or imaginary for $\mu > 0$ according as γ_{11} is negative or positive. In all cases there is at least one real solution.

The treatment of cases where there is a higher order of multiplicity of the roots $\alpha_j^{(0)}$ and more numerous congruences is similar. If the total number of roots equal to $\alpha_1^{(0)}$ is ν_1, and of those congruent to $\alpha_1^{(0)}$ mod $\sqrt{-1}$ is ν_2, then in

the expansion of D the term of lowest degree in β_1 alone is of degree $\nu_1 + \nu_2$, and the term of lowest degree in μ alone is in general of degree ν_1. There are no terms of degree lower than ν_1 in β_1 and μ.

§ 9.   *Solutions when $D = 0$ Has Two Roots Equal Identically in μ.*

The conditions necessary that $D = 0$ shall have two or more roots identical in μ are that

$$D(\alpha, \mu) = 0, \quad \frac{\partial}{\partial \alpha} D(\alpha, \mu) = 0$$

for all $|\mu|$ sufficiently small. Let us suppose, for simplicity, that α_1 and α_2 are identically equal in μ and that all the other o_j are distinct. In this case D_0 has the form (31), but there are no terms in D of the first degree in μ, for then β_{11} and β_{12} could not be equal. The value of $\beta_{\cdot 1} = \beta_{12}$ is found as in § 7, and the corresponding solution is obtained by solving (21) for the ratios of the A_j and substituting the results in (20). If all the first minors of D vanish for $\alpha = \alpha_1$, then two of the A_j remain arbitrary and we obtain in this way the two solutions belonging to α_1.

Suppose not all the first minors of D are zero for $\alpha = \alpha_1$. Then there remains but one arbitrary in the solution of (21) for the ratios of the A_j. Substituting this result in (20) we get the solution $y_{i1}(\mu, t)$, from which we have by the general transformation

$$x_{i1} = e^{\alpha_1 t} y_{i1}, \tag{35}$$

where the y_{i1} are power series in μ whose coefficients are periodic in t with the period 2π.

By the general theory of § 2 we know that the other solution depending upon α_1 is

$$x_{i2} = e^{\alpha_1 t}(y_{i2} + t y_{i1}), \qquad i = 1, \ldots, n, \tag{36}$$

where the y_{i2} are periodic in t with the period 2π. Substituting these expressions in the differential equations (17) and making use of the fact that the $e^{\alpha_1 t} y_{i1}$ are a solution, we get

$$y_{i2}' + \alpha_1 y_{i2} - \sum_{j=1}^{n} \left[a_{ij} + \sum_{k=1}^{\infty} \theta_{ij}^{(k)} \mu^k \right] y_{j2} = - y_{i1}, \qquad i = 1, \ldots, n. \tag{37}$$

If the right members of these equations are put equal to zero, they become precisely of the form of the equations satisfied by y_{i1}. Consequently the only solution of these equations which is periodic with the period 2π is $y_{i2} = y_{i1}$ plus

such particular integrals that (37) shall be satisfied when the right members are retained. The part y_{i1} is useless since it belongs to the first solution, which contains an arbitrary factor. The method of finding the particular integrals will be taken up in § 16. It will there be shown that the y_{i2} are also power series in μ.

When $D = 0$ has a multiple root of higher order of multiplicity for all $|\mu|$ sufficiently small, the y_{i1}, y_{i2}, ..., $i = 1, ..., n$, are found in succession, the problem for y_{i2}, y_{i3}, ... being that of linear equations with right members analogous to (37).

§ 10. *Direct Construction of the Solutions when the* $\alpha_i^{(0)}$ *Are Distinct and*

$$\alpha_j^{(0)} - \alpha_i^{(0)} \not\equiv 0 \ mod \ \sqrt{-1}, \quad i = 1, ..., n, \quad j = 1, ..., n.$$

The methods given above lead to the constructions of the solutions, but the work is very laborious. However, the knowledge of their properties which we have obtained and their expansibility as power series in μ lead us to convenient methods for their construction.

Under the hypotheses of this article it has been shown that there are n distinct values of α, viz. α_1, ..., α_n, expansible as power series in μ such that

$$x_{ik} = e^{\alpha_k t} y_{ik}, \quad i = 1, ..., n,$$

constitute a fundamental set of solutions, where the y_{ik} are purely periodic and expansible as power series in μ. Since the y_{ik} are expansible as converging power series in μ and are periodic with the period 2π, each $y_{ik}^{(\lambda)}$ is separately periodic. Therefore the conditions on $\alpha = \alpha_j^{(0)} + \alpha_j^{(1)}\mu + ...$ and the $y_{ik}^{(\lambda)}$ are that they shall satisfy the differential equations identically in μ and

$$y_{ik}^{(\lambda)}(t + 2\pi) - y_{ik}^{(\lambda)}(t) = 0. \tag{38}$$

The differential equations for the y_i are

$$y_i' + \alpha y_i = \sum_{j=1}^{n} \left[a_{ij} + \sum_{\lambda=1}^{\infty} \theta_{ij}^{(\lambda)} \mu^\lambda \right] y_j, \quad i = 1, ..., n. \tag{39}$$

For $\mu = 0$ the roots of the characteristic equation are $\alpha_1^{(0)}$, ..., $\alpha_n^{(0)}$. Consider any one of them, as $\alpha_k^{(0)}$. Then α and the y_i are expressible by converging power series of the form

$$\left. \begin{aligned} \alpha &= \alpha_k^{(0)} + \alpha_k^{(1)}\mu + ... = \sum_{\nu=0}^{\infty} \alpha_k^{(\nu)} \mu^\nu, \\ y_{ik} &= y_{ik}^{(0)} + y_{ik}^{(1)}\mu + ... = \sum_{\nu=0}^{\infty} y_{ik}^{(\nu)} \mu^\nu, \quad i = 1, ..., n. \end{aligned} \right\} \tag{40}$$

Substituting (40) in (39) and equating coefficients of corresponding powers of μ, we have a series of sets of differential equations from which a_k and the y_{ik} can be determined so that the y_{ik} shall be periodic with the period 2π. The determination is unique except for the arbitrary constant factor of the solution. For simplicity this arbitrary will be determined so that $y_{1k}(0) = c_{1k}$. If c_{1k} were zero, the initial condition would be imposed upon another $y_{ik}^{(0)}$, not all of which can vanish at $t = 0$. The arbitrary can be restored if desired in the final solution by multiplying by any factor not zero. We shall consider only enough steps of the process to enable us to exhibit its peculiarities and to establish its general character.

Terms Independent of μ. The terms of the solution independent of μ are defined by the differential equations

$$(y_{ik}^{(0)})' + a_k^{(0)} y_{ik}^{(0)} - \sum_{j=1}^{n} a_{ij} y_{jk}^{(0)} = 0, \qquad i = 1, \ldots, n,$$

the general solution of which is

$$y_{ik}^{(0)} = \sum_{j=1}^{n} \eta_{jk}^{(0)} c_{ij} e^{(a_j^{(0)} - a_k^{(0)})t}, \qquad i = 1, \ldots, n,$$

where the $\eta_{jk}^{(0)}$ are the constants of integration. Since the $y_{ik}^{(0)}$ are to be made periodic with the period 2π, and since $a_j^{(0)} - a_k^{(0)} \not\equiv 0 \mod \sqrt{-1}$ by hypothesis except when $j = k$, it follows that $\eta_{jk}^{(0)} = 0$ when $j \neq k$. Since $y_{1k}(0) = c_{1k}$ for all $|\mu|$ sufficiently small, it follows that $y_{1k}^{(0)}(0) = c_{1k}$, $y_{1k}^{(j)}(0) = 0$, $j = 1, \ldots, \infty$. Therefore we must make $\eta_{kk} = 1$. The solution satisfying the conditions laid down is therefore found to be

$$y_{ik}^{(0)} = c_{ik}, \qquad i = 1, \ldots, n. \tag{41}$$

Coefficients of μ. The differential equations for the coefficients of the first power of μ are

$$(y_{ik}^{(1)})' + a_k^{(0)} y_{ik}^{(1)} - \sum_{j=1}^{n} a_{ij} y_{jk}^{(1)} = -a_k^{(1)} y_{ik}^{(0)} + \sum_{j=1}^{n} \theta_{ij}^{(1)} y_{jk}^{(0)}, \qquad i = 1, \ldots, n. \tag{42}$$

When the left members of these equations are set equal to zero their general solution is

$$y_{ik}^{(1)} = \sum_{j=1}^{n} \eta_{jk}^{(1)} c_{ij} e^{(a_j^{(0)} - a_k^{(0)})t}, \qquad i = 1, \ldots, n, \tag{43}$$

where the $\eta_{jk}^{(1)}$ are the constants of integration, and the c_{ij} are the same as before.

We shall find the complete solutions of (42) by the method of the variation of parameters. Regarding the $\eta_{jk}^{(1)}$ as variables and imposing the conditions that (43) shall satisfy (42), we get

$$\sum_{j=1}^{n} (\eta_{jk}^{(1)})' c_{ij} e^{(a_j^{(0)}-a_k^{(0)})t} = - a_k^{(1)} y_{ik}^{(0)} + \sum_{j=1}^{n} \theta_{ij}^{(1)} y_{jk}^{(0)} = g_{ik}^{(1)}(t), \qquad i = 1, \ldots, n, \quad (44)$$

where the $g_{ik}^{(1)}(t)$ are periodic in t with the period 2π. The determinant of the coefficients of the $(\eta_{jk}^{(1)})'$ is

$$\Delta = |c_{ij}| e^{\sum_{j=1}^{n}(a_j^{(0)}-a_k^{(0)})t} = e^{\sum_{j=1}^{n}(a_j^{(0)}-a_k^{(0)})t},$$

which can not vanish for any finite t. Therefore the solutions of (44) for the $(\eta_{jk}^{(1)})'$ are

$$(\eta_{jk}^{(1)})' = e^{-(a_j^{(0)}-a_k^{(0)})t} \Delta_{jk}^{(1)}, \qquad (45)$$

where the $\Delta_{jk}^{(1)}$ are known periodic functions of t with the period 2π.

For $j \neq k$ the solutions of (45) have the form

$$\eta_{jk}^{(1)} = e^{-(a_j^{(0)}-a_k^{(0)})t} P_{jk}(t) + B_{jk}^{(1)}, \qquad (46)$$

where the $P_{jk}^{(1)}(t)$ are periodic with the period 2π and the $B_{jk}^{(1)}$ are arbitrary constants. For $j = k$ we have, from (44),

$$(\eta_{kk}^{(1)})' = \Delta_{kk}^{(1)} = - a_k^{(1)} + \delta_{kk}^{(1)}, \qquad (47)$$

where $\delta_{kk}^{(1)}$ is $\Delta_{kk}^{(1)}$ after the terms $- a_k^{(1)} y_{ik}^{(0)}$ have been omitted from the kth column. It is a periodic function of t with the period 2π and has in general a term independent of t. Hence we may write

$$\delta_{kk}^{(1)} = d_k^{(1)} + Q_k^{(1)}(t),$$

where $d_k^{(1)}$ is constant and the mean value of $Q_k^{(1)}(t)$ is zero. Then (47) becomes

$$(\eta_{kk}^{(1)})' = (d_k^{(1)} - a_k^{(1)}) + Q_k^{(1)}(t). \qquad (48)$$

In order that $\eta_{kk}^{(1)}$ shall be periodic we must impose the condition

$$a_k^{(1)} = d_k^{(1)}, \qquad (49)$$

after which we get

$$\eta_{kk}^{(1)} = P_{kk}^{(1)} + B_{kk}^{(1)}, \qquad (50)$$

where $P_{kk}^{(1)}$ is periodic with the period 2π and $B_{kk}^{(1)}$ is the constant of integration.

Substituting (46) and (50) in (43), we have as the general solution of (42)

$$y_{ik}^{(1)} = \sum_{j=1}^{n} B_{jk}^{(1)} c_{ij} e^{(a_j^{(0)}-a_k^{(0)})t} + \sum_{j=1}^{n} c_{ij} P_{jk}^{(1)}(t), \qquad i = 1, \ldots, n. \qquad (51)$$

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In order that the $y_{ik}^{(1)}$ shall be periodic with the period 2π, all the $B_{jk}^{(1)}$ must be zero except $B_{kk}^{(1)}$. From $y_{1k}^{(1)}(0) = 0$, we get

$$B_{kk}^{(1)} = -\frac{1}{c_{1k}} \sum_{j=1}^{n} c_{1j} P_{jk}^{(1)}(0). \tag{52}$$

Therefore the solution satisfying all the conditions is

$$y_{ik}^{(1)} = \sum_{j=1}^{n} \left[c_{ij} P_{jk}^{(1)}(t) - \frac{c_{ik}}{c_{1k}} c_{1j} P_{jk}^{(1)}(0) \right]. \tag{53}$$

It remains to be shown that the integration of the coefficients of the higher powers of μ can be effected in a similar manner. Suppose $\alpha_k^{(1)}, \ldots, \alpha_k^{(m-1)}$ and the $y_{ik}^{(1)}, y_{ik}^{(2)}, \ldots, y_{ik}^{(m-1)}$ satisfying the differential equations have been uniquely determined so that the $y_{ik}^{(l)}$ are periodic with the period 2π and that $y_{1k}^{(l)}(0) = 0$, $l = 1, \ldots, m-1$. We shall show that the $y_{ik}^{(m)}$ can be determined so as to satisfy the same conditions.

From equations (39) and (40) we find

$$(y_{ik}^{(m)})' + \alpha_k^{(0)} y_{ik}^{(m)} - \sum_{j=1}^{n} a_{ij} y_{jk}^{(m)} = -\alpha_k^m y_{ik}^{(0)} + \sum_{j=1}^{n} \theta_{ij}^{(m)} y_{jk}^{(0)}$$
$$+ \sum_{p=1}^{m-1} \left[-\alpha_k^{(p)} y_{ik}^{(m-p)} + \sum_{j=1}^{n} \theta_{ij}^{(p)} y_{jk}^{(m-p)} \right]. \tag{54}$$

Omitting the terms included under the sign of summation with respect to p, these equations are identical in form with (42) except that we now have the superscript (m) instead of (1). The integrations proceed as in the case treated, for the terms included under the summation with respect to p are all periodic with the period 2π and do not change the character of the $g_{ik}^{(m)}(t)$. Therefore $\alpha_k^{(m)}$ and the $y_{ik}^{(m)}$ can be uniquely determined so as to satisfy the differential equations and be periodic in t with the period 2π, and so that at the same time $y_{1k}^{(m)}(0) = 0$. Hence the induction is complete and the process can be indefinitely continued.

§ 11. *Construction of the Solutions when* $\alpha_2^{(0)} - \alpha_1^{(0)} \equiv 0 \mod \sqrt{-1}$.

Suppose the $\alpha_j^{(0)}$ are all distinct, that $\alpha_2^{(0)}$ and $\alpha_1^{(0)}$ differ by an imaginary integer, and that there are no other such congruences among the $\alpha_j^{(0)}$. The solutions associated with $\alpha_3^{(0)}, \ldots, \alpha_n^{(0)}$ are computed by the methods of § 10. It was shown in § 6 that in this case, in general, α_1, α_2 and the y_{i1}, y_{i2} can be developed as converging series in integral powers of μ. It will be assumed

that we are not treating one of the exceptional cases where the series proceed in fractional powers of μ.

The general solution of (39) for the terms independent of μ is in this case

$$y_{i1}^{(0)} = \sum_{j=1}^{n} \eta_{j1}^{(0)} c_{ij} e^{(a_j^{(0)} - a_1^{(0)})t}, \qquad i = 1, \ldots, n,$$

where the $\eta_{j1}^{(0)}$ are the constants of integration.

Imposing the conditions that the $y_{i1}^{(0)}$ shall be periodic with the period 2π and that $y_{11}^{(0)}(0) = c_{11}$, these equations become, since $a_2^{(0)} - a_1^{(0)}$ is an imaginary integer,

$$y_{i1}^{(0)} = \left(1 - \eta_{21}^{(0)} \frac{c_{12}}{c_{11}}\right) c_{i1} + \eta_{21}^{(0)} c_{i2} e^{(a_2^{(0)} - a_1^{(0)})t}, \qquad i = 1, \ldots, n, \tag{55}$$

where $\eta_{21}^{(0)}$ is so far arbitrary.

Coefficients of μ. It follows from (39) and (40) that the coefficients of μ must satisfy the differential equations

$$(y_{i1}^{(1)})' + a_1^{(0)} y_{i1}^{(1)} - \sum_{j=1}^{n} a_{ij} y_{j1}^{(1)} = -a_1^{(1)} y_{i1}^{(0)} + \sum_{j=1}^{n} \theta_{ij}^{(1)} y_{j1}^{(0)}, \qquad i = 1, \ldots, n. \tag{56}$$

The general solution of these equations when their right members are zero is

$$y_{i1}^{(1)} = \sum_{j=1}^{n} \eta_{j1}^{(1)} c_{ij} e^{(a_j^{(0)} - a_1^{(0)})t}, \qquad i = 1, \ldots, n. \tag{57}$$

Considering the coefficients $\eta_{j1}^{(1)}$ as functions of t and imposing the conditions that (56) shall be satisfied, we get

$$\sum_{j=1}^{n} (\eta_{j1}^{(1)})' c_{ij} e^{(a_j^{(0)} - a_1^{(0)})t} = -a_1^{(1)} y_{i1}^{(0)} + \sum_{j=1}^{n} \theta_{ij}^{(1)} y_{j1}^{(0)}, \qquad i = 1, \ldots, n.$$

Substituting the values of $y_{i1}^{(0)}$ from (55) and solving, it is found that

$$\left.\begin{aligned}
(\eta_{11}^{(1)})' &= -a_1^{(1)}\left(1 - \eta_{21}^{(0)} \frac{c_{12}}{c_{11}}\right) + \eta_{21}^{(0)} \Delta_{11}^{(1)}(t) + D_{11}^{(1)}(t), \\
(\eta_{21}^{(1)})' &= -a_1^{(1)} \eta_{21}^{(0)} + \eta_{21}^{(0)} \Delta_{21}^{(1)}(t) + D_{21}^{(1)}(t), \\
(\eta_{j1}^{(1)})' &= e^{-(a_j^{(0)} - a_1^{(0)})t} \Delta_{j1}^{(1)}(t), \qquad j = 3, \ldots, n,
\end{aligned}\right\} \tag{58}$$

where the $\Delta_{j1}^{(1)}$ and $D_{j1}^{(1)}$ are periodic functions of t with the period 2π depending upon the $\theta_{ij}^{(1)}$ and $e^{(a_j^{(0)} - a_1^{(0)})t}$. In the first two equations the undetermined constants $a_1^{(1)}$ and $\eta_{21}^{(0)}$ enter only as they are exhibited explicitly.

Equations (58) are to be integrated and the results substituted in (57). In order that the $y_{i1}^{(1)}$ shall be periodic we must impose the conditions

$$- \alpha_1^{(1)} \left(1 - \eta_{21}^{(0)} \frac{c_{12}}{c_{11}}\right) + \eta_{21}^{(0)} b_{11}^{(1)} + d_{11}^{(1)} = 0,$$

$$- \alpha_1^{(1)} \eta_{21}^{(0)} + \eta_{21}^{(0)} b_{21}^{(1)} + d_{21}^{(1)} = 0, \qquad \qquad (59)$$

$$B_{j1}^{(1)} = 0, \qquad j = 3, \ldots, n,$$

where $b_{11}^{(0)}$, $b_{21}^{(1)}$, $d_{11}^{(1)}$, $d_{21}^{(1)}$ are the constant terms of $\Delta_{11}^{(1)}$, $\Delta_{21}^{(1)}$, $D_{11}^{(1)}$ and $D_{21}^{(1)}$ respectively, and where the $B_{j1}^{(1)}$ are the constants of integration obtained with the last $n - 2$ equations. Eliminating $\eta_{21}^{(0)}$ from the first two equations, we get

$$\alpha_1^{(1)2} - \left(b_{21}^{(1)} + d_{11}^{(1)} + \frac{c_{12}}{c_{11}} d_{21}^{(1)}\right) \alpha_1^{(1)} + (b_{21}^{(1)} d_{11}^{(1)} - b_{11}^{(1)} d_{21}^{(1)}) = 0. \qquad (60)$$

There are two cases, according as the discriminant of this quadratic is not zero or is zero. In the first case, which may be regarded as the general case, the two roots for $\alpha_1^{(1)}$ are distinct, corresponding to distinct values of b_1 and b_2 given in (30) in the existence proof. It was shown there that in this case the solutions proceed according to integral powers of μ. In the second case, corresponding to $b_1 = b_2$, the character of the solutions depends upon the coefficients of terms of higher degree, and they may proceed according to powers of μ or $\pm \sqrt{\mu}$. We shall assume that the discriminant is distinct from zero and proceed to the construction of the solution.

It will be shown that after one of the two pairs of values of $\alpha_1^{(1)}$ and $\eta_{21}^{(0)}$ satisfying (59) is chosen, the solution is uniquely determined except for the arbitrary constant factor which may be introduced at the end. Integrating (58), substituting the results in (5), and determining the arbitrary constants so that the solution shall be periodic, and imposing the condition that $y_{11}^{(1)}(0) = 0$, we find

$$y_{11}^{(1)} = B_{21}^{(1)} \left[-\frac{c_{12}}{c_{11}} c_{i1} + c_{i2} e^{(\alpha_2^{(0)} - \alpha_1^{(0)}) t} \right] + \sum_{j=1}^{n} \left[c_{ij} P_{j1}^{(1)}(t) - \frac{c_{1j}}{c_{11}} c_{i1} P_{j1}^{(1)}(0) \right], \qquad (61)$$

$$i = 1, \ldots, n,$$

where $B_{21}^{(1)}$ is an undetermined constant and the $P_{j1}^{(1)}$ are entirely known periodic functions of t having the period 2π.

Coefficients of μ^2. The coefficients of μ^2 are defined by

$$(y_{11}^{(2)})' + \alpha_1^{(0)} y_{11}^{(2)} - \sum_{j=1}^{n} c_{ij} y_{j1}^{(2)} = -\alpha_1^{(2)} y_{i1}^{(0)} - \alpha_1^{(1)} y_{i1}^{(1)} + \sum_{j=1}^{n} [\theta_{ij}^{(2)} y_{j1}^{(0)} + \theta_{ij}^{(1)} y_{j1}^{(1)}], \qquad (62)$$

$$i = 1, \ldots, n.$$

The general solutions of these equations when the right members are put equal to zero are the same as (57) except that the superscripts are (2) instead of (1).

Varying the $\eta_{ji}^{(2)}$, we find for the equations corresponding to (58)

$$
\left.
\begin{aligned}
(\eta_{11}^{(2)})' &= -\alpha_1^{(2)}\left(1 - \eta_{21}^{(0)}\frac{c_{12}}{c_{11}}\right) + \alpha_1^{(1)} B_{21}^{(1)}\frac{c_{12}}{c_{11}} + B_{21}^{(1)}\Delta_{11}^{(1)}(t) + D_{11}^{(2)}(t), \\
(\eta_{21}^{(2)})' &= -\alpha_1^{(2)}\eta_{21}^{(0)} + \alpha_1^{(1)} B_{21}^{(1)} + B_{21}^{(1)}\Delta_{21}^{(1)}(t) + D_{21}^{(2)}(t), \\
(\eta_{ji}^{(2)})' &= e^{-(\alpha_j^{(0)}-\alpha_1^{(0)})t}\Delta_{ji}^{(2)}(t), \qquad j=3,\ldots,n.
\end{aligned}
\right\} \tag{63}
$$

The undetermined constants $\alpha_1^{(2)}$ and $B_{21}^{(1)}$ are written explicitly in the first two equations, and it is to be noted that $\Delta_{11}^{(1)}(t)$ and $\Delta_{21}^{(1)}(t)$ are precisely the same functions of t as those which appear in (58).

In order that when we integrate (63) and substitute the results in the equations corresponding to (57) we shall have a periodic solution, we must impose the conditions

$$
\left.
\begin{aligned}
-\left(1 - \eta_{21}^{(0)}\frac{c_{12}}{c_{11}}\right)\alpha_1^{(2)} + \left(b_{11}^{(1)} + \alpha_1^{(1)}\frac{c_{12}}{c_{11}}\right)B_{21}^{(1)} + d_{11}^{(2)} &= 0, \\
-\eta_{21}^{(0)}\alpha_1^{(2)} + (b_{21}^{(1)} - \alpha_1^{(1)})B_{21}^{(1)} + d_{21}^{(2)} &= 0, \\
B_{ji}^{(2)} &= 0, \quad j=3,\ldots,n,
\end{aligned}
\right\} \tag{64}
$$

$b_{11}^{(1)}, b_{21}^{(1)}, d_{11}^{(2)}, d_{21}^{(2)}$ being the constant terms of $\Delta_{11}^{(1)}, \Delta_{21}^{(1)}, D_{11}^{(2)}$ and $D_{21}^{(2)}$ respectively. The first two equations are linear in $\alpha_1^{(2)}$ and $B_{21}^{(1)}$ and determine these quantities uniquely, provided their determinant is not zero. The determinant is

$$
\Delta = \begin{vmatrix} -1 + \eta_{21}^{(0)}\dfrac{c_{12}}{c_{11}}, & b_{11}^{(1)} + \alpha_1^{(1)}\dfrac{c_{12}}{c_{11}} \\[2mm] -\eta_{21}^{(0)}, & b_{21}^{(1)} - \alpha_1^{(1)} \end{vmatrix} = \alpha_1^{(1)} + \eta_{21}^{(0)}\left(b_{11}^{(1)} + b_{21}^{(1)}\frac{c_{12}}{c_{11}}\right) - b_{21}^{(1)}.
$$

Eliminating $\alpha_1^{(1)}$ and $\eta_{21}^{(0)}$ by means of (60) and (59), we get

$$
\Delta = \pm\sqrt{D}, \tag{65}
$$

where D is the discriminant of (60) and is by hypothesis distinct from zero. Therefore the solution of (64) for $\alpha_1^{(2)}$ and $B_{21}^{(1)}$ is unique. The sign before $\sqrt{D}$ depends upon which of the two roots of (60) is used. Now integrating (63), substituting the results in the equations corresponding to (57), and imposing the condition that $y_{11}^{(2)}(0) = 0$, it follows that

$$
y_{i1}^{(2)} = B_{21}^{(2)}\left[-\frac{c_{12}}{c_{11}}c_{i1} + c_{i2}e^{(\alpha_2^{(0)}-\alpha_1^{(0)})t}\right] + \sum_{j=1}^{n}\left[c_{ij}P_{ji}^{(2)}(t) - \frac{c_{1j}}{c_{11}}c_{i1}P_{ji}^{(2)}(0)\right], \tag{66}
$$

$$
i=1,\ldots,n,
$$

where $B_{21}^{(2)}$ is as yet an undetermined constant.

Now consider the general step in the construction of the solution. We shall have

$$(y_{i1}^{(\nu)})' + \alpha_1^{(0)} y_{i1}^{(\nu)} - \sum_{j=1}^{n} a_{ij} y_{j1}^{(\nu)} = -\alpha_1^{(\nu)} y_{i1}^{(0)} - \alpha_1^{(1)} y_{i1}^{(\nu-1)} + F_i^{(\nu)}, \qquad i = 1, \ldots, n,$$

where the $F_i^{(\nu)}$ are known periodic functions of t. When we put the right members equal to zero, the general solutions are the same as (57) except that the superscripts are (ν). By varying the constants of integration, we get equations (63) except that the superscripts are (ν) and $(\nu - 1)$ instead of (2) and (1) respectively. The conditions for periodicity are similar to (64) and have the same determinant. Consequently at this step $\alpha_1^{(\nu)}$ and $B_{21}^{(\nu-1)}$ are uniquely determined. Hence it is evident that the process can be carried as far as is desired.

For congruences of higher order analogous methods are applicable, and in the exceptional cases to this treatment the existence proof furnishes a sure guide for the construction of the solutions.

§ 12.   *Construction of the Solutions when $\alpha_2^{(0)} = \alpha_1^{(0)}$.*

For simplicity, suppose $\alpha_2^{(0)} = \alpha_1^{(0)}$ and that there are no other equalities among the $\alpha_j^{(0)}$, and that no two of them differ by an imaginary integer. The only variations from the method of § 10 are in the construction of the solutions associated with $\alpha_1^{(0)}$. It was shown in § 7 that, except in special cases, the solutions are in powers of $\pm \sqrt{\mu}$. We shall assume that we are not treating one of the special cases. Hence we have

$$\left.\begin{array}{l} \alpha_1 = \alpha_1^{(0)} + \alpha_1^{(1)} \mu^{\frac{1}{2}} + \alpha_1^{(2)} \mu + \cdots, \\[4pt] \alpha_2 = \alpha_1^{(0)} - \alpha_1^{(1)} \mu^{\frac{1}{2}} + \alpha_1^{(2)} \mu - \cdots, \\[4pt] y_{i1} = y_{i1}^{(0)} + y_{i1}^{(1)} \mu^{\frac{1}{2}} + y_{i1}^{(2)} \mu + \cdots, \\[4pt] y_{i2} = y_{i1}^{(0)} - y_{i1}^{(1)} \mu^{\frac{1}{2}} + y_{i1}^{(2)} \mu - \cdots. \end{array}\right\} \qquad (67)$$

Terms Independent of μ. The terms independent of μ are defined by

$$(y_{i1}^{(0)})' + \alpha_1^{(0)} y_{i1}^{(0)} - \sum_{j=1}^{n} a_{ij} y_{j1}^{(0)} = 0, \qquad i = 1, \ldots, n.$$

The general solution of these equations is

$$y_{i1}^{(0)} = \eta_{i1}^{(0)} c_{i1} + \eta_{2i}^{(0)} (c_{i2} + t c_{i1}) + \sum_{j=3}^{n} \eta_{ji}^{(0)} c_{ij} e^{(\alpha_j^{(0)} - \alpha_1^{(0)}) t}.$$

Imposing the conditions that the $y_{i1}^{(0)}$ shall be periodic with the period 2π and that $y_{i1}^{(0)}(0) = c_{i1}$, we get

$$y_{i1}^{(0)} = c_{i1}, \qquad i = 1, \ldots, n. \qquad (68)$$

Coefficients of μ^i. The coefficients of μ^i are defined by

$$(y_{i1}^{(1)})' + a_1^{(0)} y_{i1}^{(1)} - \sum_{j=1}^{n} a_{ij} y_{j1}^{(1)} = -a_1^{(1)} y_{i1}^{(0)} = -a_1^{(1)} c_{i1}, \qquad i = 1, \ldots, n. \quad (69)$$

The general solution of these equations when the right members are zero is

$$y_{i1}^{(1)} = \eta_{i1}^{(1)} c_{i1} + \eta_{i21}^{(1)} (c_{i2} + t c_{i1}) + \sum_{j=3}^{n} \eta_{ji}^{(1)} c_{ij} e^{(a_j^{(0)} - a_1^{(0)}) t}, \qquad i = 1, \ldots, n. \quad (70)$$

By the variation of parameters we get

$$(\eta_{i1}^{(1)})' c_{i1} + (\eta_{21}^{(1)})' (c_{i2} + t c_{i1}) + \sum_{j=3}^{n} (\eta_{ji}^{(1)})' c_{ij} e^{(a_j^{(0)} - a_1^{(0)}) t} = -a_1^{(1)} c_{i1}, \qquad i = 1, \ldots, n.$$

Solving these equations for the $(\eta_{ji}^{(1)})'$, we find

$$(\eta_{i1}^{(1)})' = -a_1^{(1)};$$
$$(\eta_{ji}^{(1)})' = 0, \qquad j = 2, \ldots, n.$$

Consequently

$$\left. \begin{aligned} \eta_{i1}^{(1)} &= -a_1^{(1)} t + B_{i1}^{(1)}, \\ \eta_{ji}^{(1)} &= B_{ji}^{(1)}, \qquad j = 2, \ldots, n. \end{aligned} \right\} \quad (71)$$

Substituting these values of $\eta_{ji}^{(1)}$ in (70), we get

$$y_{i1}^{(1)} = [B_{i1}^{(1)} - a_1^{(1)} t] c_{i1} + B_{21}^{(1)} (c_{i2} + t c_{i1}) + \sum_{j=3}^{n} B_{ji}^{(1)} c_{ij} e^{(a_j^{(0)} - a_1^{(0)}) t}, \qquad i = 1, \ldots, n. \quad (72)$$

Imposing the conditions that the $y_{i1}^{(1)}$ shall be periodic with the period 2π and that $y_{i1}^{(1)}(0) = 0$, we have

$$\left. \begin{aligned} B_{21}^{(1)} &= a_1^{(1)}, \\ B_{ji}^{(1)} &= 0, \qquad j = 3, \ldots, n, \\ B_{i1}^{(1)} c_{i1} + B_{21}^{(1)} c_{i2} &= 0. \end{aligned} \right\} \quad (73)$$

Then equations (72) become

$$y_{i1}^{(1)} = \left(-\frac{c_{i2}}{c_{i1}} c_{i1} + c_{i2} \right) a_1^{(1)}, \qquad i = 1, \ldots, n, \quad (74)$$

where $a_1^{(1)}$ remains as yet undetermined.

Coefficients of μ. The coefficients of μ satisfy

$$(y_{i1}^{(2)})' + a_1^{(0)} y_{i1}^{(2)} - \sum_{j=1}^{n} a_{ij} y_{j1} = -a_1^{(2)} y_{i1}^{(0)} - a_1^{(1)} y_{i1}^{(1)} + \sum_{j=1}^{n} \theta_{ij}^{(1)} y_{j1}^{(0)}, \qquad i = 1, \ldots, n. \quad (75)$$

The solution of these equations when the right members are zero is of the same form as (70), and we find, by varying the constants,

$$(\eta_{i1}^{(2)})' c_{i1} + (\eta_{21}^{(2)})' (c_{i2} + t c_{i1}) + \sum_{j=3}^{n} (\eta_{ji}^{(2)})' c_{ij} e^{(a_j^{(0)} - a_1^{(0)}) t} = -a_1^{(2)} y_{i1}^{(0)} - a_1^{(1)} y_{i1}^{(1)} + \sum_{j=1}^{n} \theta_{ij}^{(1)} y_{j1}^{(0)}.$$

Solving these equations, we get

$$
\begin{aligned}
(\eta^{(2)}_{11})' &= -\alpha^{(2)}_1 + \Big(\frac{c_{12}}{c_{11}} + t\Big)\alpha^{(1)^2}_1 + t\Delta^{(2)}_{11}(t) + D^{(2)}_{11}(t), \\
(\eta^{(2)}_{21})' &= \qquad\qquad -\alpha^{(1)^2}_1 - \Delta^{(2)}_{11}(t), \\
(\eta^{(2)}_{j1})' &= e^{-(\alpha^{(0)}_j - \alpha^{(0)}_1)t}\,\Delta^{(2)}_{j1}(t), \qquad j = 3, \ldots, n,
\end{aligned}
\right\} \tag{76}
$$

where $\Delta^{(2)}_{j1}(t)$ and $D^{(2)}_{11}(t)$ are periodic functions of t. The first of these equations gives rise to integrals of the type

$$
a_j \int t \frac{\sin}{\cos} jt \, dt = \mp \frac{a_j}{j} t \frac{\cos}{\sin} jt + \frac{a_j}{j^2} \frac{\sin}{\cos} jt.
$$

The second equation gives rise to the corresponding integral

$$
- a_j \int \frac{\sin}{\cos} jt \, dt = \pm \frac{a_j}{j} \frac{\cos}{\sin} jt.
$$

When we substitute these results in the equations corresponding to (70), we get for these terms

$$
(\eta^{(2)}_{11} + \eta^{(2)}_{21} t) c_{i1} = \frac{a_j}{j} \Big[\mp t \frac{\cos}{\sin} jt \pm t \frac{\cos}{\sin} jt \Big] + \frac{a_j}{j^2} \frac{\sin}{\cos} jt.
$$

Consequently the terms of the type $t \dfrac{\cos}{\sin} jt$ vanish when the results of the integration of (76) are substituted back in the equations corresponding to (70). Hence, we get at this step

$$
\begin{aligned}
y^{(2)}_{i1} &= B^{(2)}_{11} c_{i1} + B^{(2)}_{21}(c_{i2} + t c_{i1}) + \sum_{j=3}^{n} B^{(2)}_{j1} c_{ij} e^{(\alpha^{(0)}_j - \alpha^{(0)}_1)t} \\
&\quad + \Big[\Big(-\alpha^{(2)}_1 + \frac{c_{12}}{c_{11}}\alpha^{(1)^2}_1 + d^{(2)}_{11}\Big)t + \tfrac{1}{2}(\alpha^{(1)^2}_1 + b^{(2)}_{11})t^2 + P^{(2)}_{11}(t)\Big] c_{i1} \\
&\quad + \big[- (\alpha^{(1)^2}_1 + b^{(2)}_{11})t + P^{(2)}_{21}(t)\big] c_{i2} + \sum_{j=3}^{n} c_{ij} P^{(2)}_{j1}(t),
\end{aligned}
\right\} \tag{77}
$$

where $b^{(2)}_{11}$ and $d^{(2)}_{11}$ are the constant terms in $\Delta^{(2)}_{11}$ and $D^{(2)}_{11}$, and where the $P^{(2)}_{j1}(t)$ are periodic functions of t. In order that the $y^{(2)}_{i1}$ shall be periodic and $y^{(2)}_{i1}(0) = 0$, we must impose the conditions

$$
\begin{aligned}
\alpha^{(1)}_1 &= \pm \sqrt{-b^{(2)}_{111}}, \\
B^{(2)}_{21} &= \alpha^{(2)}_1 - \frac{c_{12}}{c_{11}} b^{(2)}_{11} - d^{(2)}_{11}, \\
B^{(2)}_{j1} &= 0, \qquad j = 3, \ldots, n, \\
B^{(2)}_{11} c_{11} &+ B^{(2)}_{21} c_{12} + \sum_{j=1}^{n} P^{(2)}_{j1}(0) c_{1j} = 0.
\end{aligned}
\right\} \tag{78}
$$

The only undetermined constant remaining in (77) is $\alpha_1^{(2)}$, and the $y_{i1}^{(2)}$ now have the form

$$y_{i1}^{(2)} = \left(c_{i2} - \frac{c_{12}}{c_{11}} c_{i1}\right) \alpha_1^{(2)} + \Phi_{i1}^{(2)}(t), \qquad i = 1, \ldots, n, \qquad (79)$$

where the $\Phi_{i1}^{(2)}(t)$ are known periodic functions of t.

The constant $\alpha_1^{(1)}$ has a double determination except in the special case where $b_{11}^{(2)} = 0$. We shall suppose $b_{11}^{(2)} \neq 0$, for if it is zero we have one of the special cases excluded above. It will be shown that the work becomes unique after one of the two possible values of $\alpha_1^{(1)}$ is chosen.

The coefficients of μ^3 are determined by

$$(y_{i1}^{(3)})' + \alpha_1^{(0)} y_{i1}^{(3)} - \sum_{j=1}^{n} a_{ij} y_{j1}^{(3)} = - \alpha_1^{(3)} y_{i1}^{(0)} - \alpha_1^{(2)} y_{i1}^{(1)} - \alpha_1^{(1)} y_{i1}^{(2)} + \sum_{j=1}^{n} \theta_{ij}^{(1)} y_{j1}^{(1)}.$$

The equations corresponding to (76) are

$$(\eta_{11}^{(3)})' = - \alpha_1^{(3)} + 2\left(\frac{c_{12}}{c_{11}} + t\right) \alpha_1^{(1)} \alpha_1^{(2)} + t \Delta_{11}^{(3)}(t) + D_{11}^{(3)}(t),$$

$$(\eta_{21}^{(3)})' = \qquad\qquad - 2\alpha_1^{(1)} \alpha_1^{(2)} - \Delta_{11}^{(3)}(t),$$

$$(\eta_{j1}^{(3)})' = e^{-(\alpha_j^{(0)} - \alpha_1^{(0)})t} \Delta_{j1}^{(3)}, \qquad j = 3, \ldots, n.$$

In order that the final solutions at this step shall be periodic, we must impose the condition

$$2\alpha_1^{(1)} \alpha_1^{(2)} + b_{11}^{(3)} = 0,$$

which determines $\alpha_1^{(2)}$ uniquely since $\alpha_1^{(1)} \neq 0$. The other constants are all uniquely determined by the periodicity condition and the initial condition except $\alpha_1^{(3)}$, which is fixed by the periodicity condition at the next step. Similarly, another solution is obtained using the other determination of $\alpha_1^{(1)}$. The solutions associated with $\alpha_3^{(0)}, \ldots, \alpha_n^{(0)}$ are obtained by the method of § 10.

The chief types of cases have been treated, and the exceptions to them are developed similarly, according to the forms indicated by the existence proofs.

§ 13. *Solutions when the θ_{ij} do not All Reduce to Constants for $\mu = 0$.*

Heretofore we have considered linear differential equations whose coefficients become constants when $\mu = 0$. We shall now waive this restriction, but we shall suppose the solutions are known for $\mu = 0$. Let the equations under consideration be

$$x_i' = \sum_{j=1}^{n} \left[\sum_{k=0}^{\infty} \theta_{ij}^{(k)}(t) \mu^k \right] x_j, \qquad i = 1, \ldots, n, \qquad (80)$$

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where the $\theta_{ij}^{(k)}$ are periodic functions of t with the period 2π, and where not all the $\theta_{ij}^{(0)}$ are constants. For $\mu = 0$ these equations reduce to

$$(x_i^{(0)})' = \sum_{j=1}^{n} \theta_{ij}^{(0)} x_j^{(0)}, \qquad i = 1, \ldots, n. \tag{81}$$

The solutions of these equations in general have the form

$$x_i^{(0)} = \sum_{j=1}^{n} A_j^{(0)} e^{\alpha_j^{(0)}(t)} y_{ij}^{(0)}(t), \qquad i = 1, \ldots, n, \tag{82}$$

where the $y_{ij}^{(0)}$ are periodic functions of t with the period 2π.

Suppose the $\alpha_j^{(0)}$ and $y_{ij}^{(0)}$ are fully known. We desire the solutions of (80). It follows from the general results of § 3 that there is at least one solution of the form

$$x_i = e^{\alpha t} y_i, \qquad i = 1, \ldots, n, \tag{83}$$

where the y_i are periodic with the period 2π. Now equations (80) can be integrated as power series in μ, reducing to $x_i^{(0)}$ for $\mu = 0$. We form n solutions, $x_{i1}, \ldots, x_{in}$, defined by the initial conditions $x_{i\kappa}^{(0)} = 0$, $x_{\kappa\kappa}^{(0)} = 1$, $\kappa = 1, \ldots, n$. Then any solution can be expressed linearly and homogeneously in terms of these n solutions. Hence

$$x_i = \sum_{j=1}^{n} A_j x_{ij}, \qquad i = 1, \ldots, n. \tag{84}$$

Transforming (80) by (83), we get

$$y_i' + \alpha y_i = \sum_{j=1}^{n} \left[\sum_{k=0}^{\infty} \theta_{ij}^{(k)} \mu^k \right] y_j, \qquad i = 1, \ldots, n.$$

Consequently the conditions that the y_i shall be periodic with the period 2π are $y_i(2\pi) - y_i(0) = 0$, $i = 1, \ldots, n$; or, because of (83), (84) and the initial conditions imposed on the x_{ij},

$$\sum_{j=1}^{n} A_j \left[e^{-2\alpha\pi} x_{ij}(2\pi) - x_{ij}(0) \right] = 0, \qquad i = 1, \ldots, n. \tag{85}$$

Now, since $x_{ij} = \sum_{k=0}^{\infty} x_{ij}^{(k)} \mu^k$, the determinant of the coefficients of the A_j set equal to zero is

$$\Delta = \left| e^{-2\alpha\pi} x_{ij}^{(0)}(2\pi) - x_{ij}^{(0)}(0) + e^{-2\alpha\pi} \sum_{k=1}^{\infty} x_{ij}^{(k)}(2\pi) \mu^k \right| = 0. \tag{86}$$

This is an equation for the determination of α. For $\mu = 0$ the solutions are $\alpha = \alpha_j^{(0)} + \nu \sqrt{-1}$, where ν is any integer. Consequently, if $\alpha_\kappa^{(0)}$ is a simple

root of (86) and if no two of the $\alpha_j^{(0)}$ differ by an imaginary integer, then (86) can be solved for $(\alpha - \alpha_\kappa^{(0)})$ as a converging power series in μ. The results substituted in (85) give the ratios of the A_j as power series in μ for $\kappa = 1, \ldots, n$. When these results are substituted in (84), we have the solutions expanded as converging power series in μ, and they have the form (83), where α and the y_i are power series in μ, and where the latter are periodic with the period 2π.

There are other cases where for $\mu = 0$ the roots of (86) satisfy different conditions. The discussion is parallel to that where for $\mu = 0$ the θ_{ij} are constants. The essentials are that the differential equations shall be expansible as power series in a parameter μ, and that for $\mu = 0$ the solutions shall be known. The process is fundamentally one of analytic continuation of the solutions with respect to the parameter μ, and it can be repeated and continued from one value of μ to any other, provided the series do not pass through a singularity in the interval.

NON-HOMOGENEOUS LINEAR DIFFERENTIAL EQUATIONS.[*]

§ 14. *Case where the Right Members Are Periodic with the Period 2π and the α_j Are Distinct.*

The problems of celestial mechanics generally lead to sets of differential equations having right members which are functions of the independent variable alone. The character of the terms in the solutions introduced by these right members depends not only upon the properties of the latter but also upon the α_j. We shall treat the most useful cases.

Suppose

$$x_i' - \sum_{j=1}^{n} \theta_{ij} x_j = g_i(t), \qquad i = 1, \ldots, n, \tag{87}$$

where the θ_{ij} and g_i are finite and continuous periodic functions of t having the period 2π. For the left members set equal to zero the general solutions are in this case

$$x_i = \sum_{j=1}^{n} \eta_j e^{\alpha_j t} y_{ij}, \qquad i = 1, \ldots, n, \tag{88}$$

where the y_{ij} are periodic with the period 2π and the η_j are the constants of integration.

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[*] For a different treatment of this subject see paper by W. D. MACMILLAN, *Transactions of American Mathematical Society*, Vol. XI, No. 1, p. 85.

By the method of the variation of parameters we find

$$\sum_{j=1}^{n} \eta_j' e^{a_j t} y_{ij} = g_i(t), \qquad i = 1, \ldots, n. \tag{89}$$

The determinant of the coefficients of the η_j' is the determinant of the fundamental set of solutions, and can vanish for no value of t for which the θ_{ij} are regular (§ 2). By hypothesis the θ_{ij} are regular for all finite values of t. Therefore this determinant can not vanish, and is

$$e^{\sum_{j=1}^{n} a_j t} \Delta,$$

where Δ is the determinant of the y_{ij}. Consequently the solutions of (89) have the form

$$\eta_j' = e^{-a_j t} \frac{\Delta_j}{\Delta}, \tag{90}$$

where Δ_j and Δ are periodic functions of t with the period 2π, and moreover Δ vanishes for no finite value of t. Hence Δ_j/Δ can be expanded as Fourier series of the form

$$\frac{\Delta_j}{\Delta} = a_0^{(j)} + \sum_{m=1}^{\infty} \left[a_m^{(j)} \cos mt + b_m^{(j)} \sin mt \right].$$

Since the Δ_j and Δ are power series in μ and in general Δ does not vanish for $\mu = 0$, this result can also be arranged as a power series in μ whose coefficients are periodic with the period 2π, but it is more convenient here to regard it simply as a Fourier series.

If $a_j^2 + m^2 \neq 0$, $j = 1, \ldots, n$, $m = 0, \ldots, \infty$, we have

$$\eta_j = \int e^{-a_j t} \frac{\Delta_j}{\Delta} dt = -\frac{a_0^{(j)}}{a_j} e^{-a_j t} + e^{-a_j t} \sum_{m=1}^{\infty} \left[-\frac{a_j a_m^{(j)} + m b_m^{(j)}}{a_j^2 + m^2} \cos mt \right.$$

$$\left. + \frac{m a_m^{(j)} - a_j b_m^{(j)}}{a_j^2 + m^2} \sin mt \right] + B_j;$$

or,

$$\eta_j = B_j + e^{-a_j t} P_j(t), \tag{91}$$

where the $P_j(t)$ are periodic with the period 2π and the B_j are constants of integration. Substituting in (88), we get

$$x_i = \sum_{j=1}^{n} B_j e^{a_j t} y_{ij} + \sum_{j=1}^{n} y_{ij} P_j(t), \qquad i = 1, \ldots, n, \tag{92}$$

as the general solutions of (87), the $P_j(t)$ being periodic with the period 2π. The terms $y_{ij} P_j(t)$ are the ones due to the presence of the $g_i(t)$.

Now let us suppose that $\alpha_k = \nu \sqrt{-1}$, where ν is an integer. Then the term

$$\int e^{-\nu \sqrt{-1}\,t} \left[a_\nu \cos \nu t + b_\nu \sin \nu t \right] dt$$

becomes

$$\tfrac{1}{2} (a_\nu - b_\nu \sqrt{-1})\, t + \frac{1}{4\nu} (a_\nu \sqrt{-1} - b_\nu)(\cos 2\nu t - \sqrt{-1} \sin 2\nu t).$$

Hence in this case we get

$$x_i = \sum_{j=1}^{n} B_j e^{\alpha_j t} y_{ij} + \sum_{j=1}^{n} y_{ij}\, P_j(t) + \tfrac{1}{2}(a_\nu - b_\nu \sqrt{-1})\, t\, e^{\alpha_\kappa t} y_{i\kappa}, \quad i = 1, \ldots, n. \quad (93)$$

Hence, *if the α_j are distinct and none of them congruent to zero mod $\sqrt{-1}$, and if the $g_i(t)$ are periodic with the period 2π, then the particular integrals are also periodic with the period 2π; but if one of the characteristic exponents is congruent to zero mod $\sqrt{-1}$, then the particular integrals in general will contain, in addition to periodic terms, the corresponding parts of the complementary function multiplied by a constant times t.*

§ 15. *Case where the Right Members Are Periodic Terms Multiplied by an Exponential, and the α_j Are Distinct.*

We now suppose the $g_i(t)$ have the form

$$g_i(t) = e^{\lambda t} f_i(t), \quad (94)$$

where the $f_i(t)$ are periodic with the period 2π. When λ is a pure imaginary, $\lambda = l \sqrt{-1}$, the $g_i(t)$ have the form

$$g_i(t) = \sum_{k=0}^{\infty} \left[a_k \cos (k+l)\, t + b_k \sin (k+l)\, t \right],$$

which appears often in applications to celestial mechanics.

If we transform the differential equations

$$x_i - \sum_{j=1}^{n} \theta_{ij} x_j = e^{\lambda t} f_i(t) \quad (95)$$

by

$$x_i = e^{\lambda t} z_i,$$

we obtain

$$z_i' + \lambda z_i - \sum_{j=1}^{n} \theta_{ij} z_i = f_i(t), \quad i = 1, \ldots, n, \quad (96)$$

which have the same character as the equations treated in § 14. If the α_j are distinct, then the characteristic exponents $\alpha_j - \lambda$ belonging to (96) are also

distinct. If no $\alpha_j - \lambda$ is congruent to zerc mod $\sqrt{-1}$, then the general solutions of (96) are

$$z_i = B_j e^{(\alpha_j - \lambda)t} y_{ij} + Q_i(t),$$

where the $Q_i(t)$ are periodic with the period 2π.

Therefore in this case the general solutions of (95) are

$$x_i = \sum_{j=1}^{n} B_j e^{\alpha_j t} y_{ij} + e^{\lambda t} Q_i(t), \qquad i = 1, \ldots, n. \tag{97}$$

But if λ is congruent to one of the α_j, say α_ν, mod $\sqrt{-1}$, then the z_i have the form

$$z_i = \sum_{j=1}^{n} B_j e^{(\alpha_j - \lambda)t} y_{ij} + \sum_{j=1}^{n} y_{ij} P_j(t) + \tfrac{1}{2}(c_\nu - b_\nu \sqrt{-1}) t e^{(\alpha_\nu - \lambda)t} y_{i\nu},$$

and therefore

$$x_i = \sum_{j=1}^{n} B_j e^{\alpha_j t} y_{ij} + e^{\lambda t} \sum_{j=1}^{n} y_{ij} P_j(t) + \tfrac{1}{2}(a_\nu - b_\nu \sqrt{-1}) t e^{\alpha_\nu t} y_{i\nu}. \tag{98}$$

Therefore, *if the $g_i(t)$ are $e^{\lambda t}$ times periodic functions and if no $\alpha_j - \lambda$ is congruent to zero mod $\sqrt{-1}$, then the particular solution is $e^{\lambda t}$ times a periodic function; but if $\alpha_\nu - \lambda$ is congruent to zero mod $\sqrt{-1}$, then the particular solution is $e^{\lambda t}$ times a periodic function plus a constant times $t e^{\alpha_\nu t} y_i$.*

§ 16. *Case where the Right Members Are Periodic and $\alpha_2 = \alpha_1$.*

In case there are no equalities among the α_j except $\alpha_2 = \alpha_1$, the solutions of

$$x_i' - \sum_{j=1}^{n} \theta_{ij} x_j = 0, \qquad i = 1, \ldots, n,$$

in general have the form

$$x_i = \eta_1 e^{\alpha_1 t} y_{i1} + \eta_2 e^{\alpha_1 t} (y_{i2} + t y_{i1}) + \sum_{j=3}^{n} \eta_j e^{\alpha_j t} y_{ij}, \qquad i = 1, \ldots, n. \tag{99}$$

For the non-homogeneous equations

$$x_i' - \sum_{j=1}^{n} \theta_{ij} x_j = g_i(t)$$

we find, by the variation of parameters,

$$e^{\alpha_1 t} y_{i1} \eta_1' + e^{\alpha_1 t} (y_{i2} + t y_{i1}) \eta_2' + \sum_{j=3}^{n} e^{\alpha_j t} y_{ij} \eta_j' = g_i(t), \qquad i = 1, \ldots, n.$$

Solving these equations for the η_j', we get

$$\Delta\eta_1' = |\ g_i(t),\ \ (y_{i2} + t y_{i1}),\ \ y_{i3},\ \ \ldots,\ \ y_{in}\ |\ e^{-\alpha_1 t},$$

$$\Delta\eta_2' = |\ y_{i1},\ \ \ g_i(t),\ \ \ \ y_{i3},\ \ \ldots,\ \ y_{in}\ |\ e^{-\alpha_1 t},$$

$$\Delta\eta_j' = |\ y_{i1},\ \ (y_{i2} + t y_{i1}),\ \ y_{i3},\ \ \ldots,\ \ y_{in}\ |\ e^{-\alpha_j t},\ \ \ j = 3,\ \ldots, n,$$

where Δ is the determinant $|y_{ij}|$. The expansions of these determinants have the form

$$\left.\begin{aligned}\eta_1' &= e^{-\alpha_1 t} P_1(t) - e^{-\alpha_1 t} t\, P_2(t), \\ \eta_2' &= e^{-\alpha_1 t} P_2(t), \\ \eta_j' &= e^{-\alpha_j t} P_j(t), \qquad j = 3,\ \ldots, n,\end{aligned}\right\} \tag{100}$$

where the $P_1(t),\ \ldots, P_n(t)$ are periodic with the period 2π.

If no α_j is congruent to zero mod $\sqrt{-1}$, then we find

$$\left.\begin{aligned}\eta_1 &= e^{-\alpha_1 t} R_1(t) - e^{-\alpha_1 t} t\, R_2(t) + B_1, \\ \eta &= e^{-\alpha_1 t} R_2(t) + B_2, \\ \eta_j &= e^{-\alpha_j t} R_j(t) + B_j, \qquad j = 3,\ \ldots, n,\end{aligned}\right\} \tag{101}$$

where $R_1,\ \ldots, R_n$ are periodic with the period 2π. Substituting (101) in (99), we get

$$x_i = B_1 e^{\alpha_1 t} y_{i1} + B_2 e^{\alpha_1 t} (y_{i2} + t y_{i1}) + \sum_{j=3}^{n} B_j e^{\alpha_j t} y_{ij} + \sum_{j=1}^{n} R_j y_{ij}. \tag{102}$$

The terms introduced into the solutions by the right members of the differential equations are $\sum\limits_{j=1}^{n} R_j y_{ij}$, which are periodic. Therefore *it follows that equalities among the characteristic exponents, without congruences to zero mod $\sqrt{-1}$, do not introduce non-periodic terms into this part of the solutions.*

The case where one α_j, $j = 3,\ \ldots, n$, is congruent to zero mod $\sqrt{-1}$, is a combination of the second part of §14 and this case; and that where $\alpha_2 = \alpha$ is congruent to zero mod $\sqrt{-1}$ does not differ from that where $\alpha_2 = \alpha_1 = 0$.

Consequently we consider the case $\alpha_2 = \alpha_1 = 0$, which frequently arises in celestial mechanics. Then the equations corresponding to (100) become

$$\left.\begin{aligned}\eta_1' &= P_1(t) - t\, P_2(t), \\ \eta_2' &= P_2(t), \\ \eta_j' &= e^{-\alpha_j t} P_j(t), \qquad j = 3,\ \ldots, n,\end{aligned}\right\} \tag{103}$$

where

$$P_j = \sum_{k=0}^{\infty} [a_k^{(j)} \cos kt + b_k^{(j)} \sin kt], \qquad j = 1,\ \ldots, n.$$

Hence we find

$$\left.\begin{aligned}
\eta_1 &= a_0^{(1)} t - \tfrac{1}{2} a_0^{(2)} t^2 - t R_2(t) + R_1(t) + B_1, \\
\eta_2 &= a_0^{(2)} t + R_2(t) + B_2, \\
\eta_j &= e^{-\alpha_j t} R_j(t) + B_j, \qquad j = 3, \ldots, n,
\end{aligned}\right\} \quad (104)$$

where the $R_j(t)$ are periodic with the period 2π. Substituting these values in (99), we get

$$x_i = B_1 y_{i1} + B_2 (y_{i2} + t y_{i1}) + \sum_{j=1}^{n} B_j e^{\alpha_j t} y_{ij} + (a_0^{(1)} t + \tfrac{1}{2} a_0^{(2)} t^2) y_{i1}$$
$$+ a_0^{(2)} t y_{i2} + \sum_{j=3}^{n} R_j(t) y_{ij}. \qquad (105)$$

Hence, *when the* $g_i(t)$ *are periodic with the period* 2π*, and two of the* α_j *are not only equal but also equal to zero, then the particular integral involves not only* t *but, in general, also* t^2 *outside of the trigonometric symbols.* It can be shown similarly that when k of the α_j are equal to zero, then in general the solutions are polynomials in t of degree k whose coefficients are periodic functions of t.

On Three-Spreads Satisfying Four or More Homogeneous Linear Partial Differential Equations of the Second Order.

By Charles H. Sisam.

Introduction.

1. In an article in the *Atti della Accademia Reale delle Scienze di Torino*,[*] Segre discussed the two-spreads which satisfy one or more homogeneous linear partial differential equations of the second order. The discussion here given follows the same order of ideas.

The equations of the three-spread are supposed to be given in the parametric form:

$$x_i = f_i(u_1, u_2, u_3), \qquad i = 0, 1, \ldots, n.$$

The three-spread will be said to satisfy an homogeneous partial differential equation when each of the $n + 1$ functions f_i satisfies the equation.

2. It will first be determined under what conditions a three-spread may satisfy more than four homogeneous linear partial differential equations of the second order. It will next be shown that, if the three-spread satisfies four such equations, it has, at an arbitrary point, four tangents having contact of the second order with the three-spread. It will then be determined under what conditions two or more of these three-point tangents, at an arbitrary point, may be consecutive.

The functions f_i are, in general, supposed to be analytic, although, in a large part of the work, this is a greater restriction than is necessary.

Differentiation will be denoted by indices, for example:

$$\frac{\partial f}{\partial u_1}, \quad \frac{\partial^2 f}{\partial u_3^2}, \quad \frac{\partial^3 f}{\partial u_1 \partial u_2 \partial u}$$

will be denoted, respectively, by f^1, f^{33} and f^{123}.

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[*] "Su una Classe di Superficie degl' Iperspazii Legate colle Equazioni Lineari alle Derivate Parziali di 2° Ordine." Vol. XLII (1907), p. 559.

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3.   Let $f_0 \neq 0$ in the region under consideration and let

$$g_i = \frac{f_i}{f_0}, \qquad i = 1, 2, \ldots, n.$$

Then the equations of the three-spread may be written non-homogeneously in the form:

$$x_i = g_i(u_1, u_2, u_3), \qquad i = 1, 2, \ldots, n.$$

Since the entity under consideration is, by hypothesis, a three-spread, the matrix,

$$\left\| \frac{\partial g_i}{\partial u_j} \right\|, \qquad \begin{array}{l} i = 1, 2, \ldots, n, \\ j = 1, 2, 3, \end{array} \qquad (1)$$

must be of rank three.

4.   It follows that the functions f_i do not all satisfy an homogeneous linear partial differential equation of the first order:

$$a_1 f^1 + a_2 f^2 + a_3 f'^3 + af = 0.$$

For, suppose that such an equation were satisfied. It would then follow that the functions $g_i = \dfrac{f_i}{f_0}$ all satisfy

$$a_1 g^1 + a_2 g^2 + a_3 g^3 = 0.$$

Hence, since the matrix (1) is of rank three, it follows that $a_1 = a_2 = a_3 = 0$. Since $f_0 \neq 0$, it now follows that $a = 0$.

On Three-Spreads Satisfying More Than Four Homogeneous Linear Partial Differential Equations of the Second Order.

5.   No three-spread can satisfy more than six independent homogeneous linear partial differential equations of the second order. For, if it could satisfy seven such equations, it would have to satisfy the equation of the first order, which could be deduced from them. This, we have seen, is impossible.

6.   If the three-spread satisfies six such equations, these equations must be reducible to the form

$$f^{hk} = a_{hk}f + b_{hk}f^1 + c_{hk}f^2 + d_{hk}f^3, \qquad h = 1, 2, 3; \quad k = 1, 2, 3, \qquad (1)$$

since, otherwise, from the six equations could be deduced one of the first order.

A three-spread which satisfies six equations of the above form lies in an S_8.*

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*   The notation S_r will be used to denote a space of r dimensions.

For, let $f_0 \neq 0$ in the region considered and let $g_i = \dfrac{f_i}{f_0}$. From the above six equations we obtain:

$$g^{hk} = \beta_{hk} g^1 + \gamma_{hk} g^2 + \delta_{hk} g^3, \qquad h = 1, 2, 3; \quad k = 1, 2, 3. \qquad (2)$$

Since the matrix $\left\| \dfrac{\partial g_i}{\partial u_j} \right\|$ is of rank three, there exists at least one determinant of third order in the matrix which is not zero. Let

$$\begin{vmatrix} g_1^1 & g_2^1 & g_3^1 \\ g_1^2 & g_2^2 & g_3^2 \\ g_1^3 & g_2^3 & g_3^3 \end{vmatrix} \neq 0.$$

If we now take g_1, g_2 and g_3 for independent variables, it follows from equations (2) that

$$\frac{\partial^2 g_i}{\partial g_h \partial g_k} = 0, \qquad i = 1, 2, \ldots, n; \quad h = 1, 2, 3; \quad k = 1, 2, 3.$$

Hence,

$$g_i = a_i g_1 + b_i g_2 + c_i g_3 + d_i, \qquad i = 1, 2, \ldots, n,$$

where a_i, b_i, etc., are constants. Hence, the three-spread lies in an S_3.

7. Through each point of a three-spread which satisfies *five* independent homogeneous linear partial differential equations of the second order pass an infinite number of tangents having contact of the second order with the three-spread. For, let

$$\Omega_i = \sum_{h,\,k=1}^{3} \omega_{hk} f_i^{hk}.$$

By a suitable choice of the quantities ω_{hk}, it is now possible to replace the five given equations by the following six:

$$f^{hk} = a_{hk} f + b_{hk} f^1 + c_{hk} f^2 + d_{hk} f^3 + e_{hk} \Omega, \qquad h = 1, 2, 3; \quad k = 1, 2, 3. \qquad (3)$$

The three-point tangents at (u_1, u_2, u_3) are those determined by values of the ratios $du_1 : du_2 : du_3$ which reduce to four the rank of the matrix:

$$\begin{Vmatrix} f_0 & f_1 & \cdots & f_n \\ f_0^1 & f_1^1 & \cdots & f_n^1 \\ f_0^2 & f_1^2 & \cdots & f_n^2 \\ f_0^3 & f_1^3 & \cdots & f_n^3 \\ \varphi_0 & \varphi_1 & \cdots & \varphi_n \end{Vmatrix}, \qquad (4)$$

in which

$$\varphi_i = \sum_{h=1}^{3} \sum_{k=1}^{3} f_i^{hk} \, du_h \, du_k, \qquad i = 0, 1, \ldots, n.$$

Substituting for the functions f_i^{hk} their values from equations (3) and subtracting multiples of the first four rows of the matrix, it is found that ϕ_i may be replaced by θ_i, where

$$\theta_t = \Omega_i \sum_{h=1}^{3} \sum_{k=1}^{3} e_{hk} \, du_h \, du_k, \qquad i = 0, 1, \ldots, n.$$

Since, for all values of i, θ_i contains the factor

$$\sum_{h=1}^{3} \sum_{k=1}^{3} e_{hk} \, du_h \, du_k, \tag{5}$$

any set of values of $du_1 : du_2 : du_3$ which annuls this factor reduces the rank of the matrix to four. Hence, any tangent at (u_1, u_2, u_3) determined by such a set of values of $du_1 : du_2 : du_3$ is a three-point tangent at (u_1, u_2, u_3) to the three-spread.

8. Since the factor (5) is quadratic in (du_1, du_2, du_3), the three-point tangents at each point (u_1, u_2, u_3) of the three-spread form a quadric cone. It follows that the three-spread is either an hypersurface in S_4 or is formed by a system of planes such that consecutive planes intersect in a line.

For, the section of the three-spread by an arbitrary hyperplane is a two-spread having, at each point, two three-point tangents. SEGRE has shown[*] that such a two-spread either lies in an S_3 or is a developable or cone.

9. If the section of the three-spread by an arbitrary hyperplane is a two-spread lying in an S_3, then the three-spread lies in an S_4. For, if not, let P_1, $P_2, \ldots, P_6$ be six points of the three-spread which determine an S_5. An hyperplane passing through $P_1, P_2, \ldots, P_5$, and not passing through P_6, intersects the three-spread in a two-spread not lying in an S_3.

10. If the two-spread of section by an arbitrary hyperplane is a developable or cone, the two three-point tangents at each point of the two-spread are consecutive and lie entirely on the two-spread. Hence, the cone of three-point tangents to the three-spread at an arbitrary point degenerates into a double plane lying entirely on the three-spread. The three-spread is therefore generated by a simply infinite system of planes. Consecutive planes of this system intersect in a line, since the system of planes is intersected by an arbitrary hyperplane in a system of lines such that consecutive lines intersect. Such a three-spread is generated by either (a) the osculating planes to a curve, or

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(b) the planes projecting the tangents to a curve from a fixed point, or (c) the planes projecting the points of a curve from a fixed line.

11. Conversely, let

$$x_i = f_i(u_1, u_2, u_3), \qquad i = 0, 1, 2, 3, 4,$$

be the equations of an hypersurface in an S_4. The condition that these five functions f_i satisfy an homogeneous linear partial differential equation of the second order is only five conditions on the ten coefficients of the differential equation. There exist, therefore, five such equations satisfied by each of the five functions. Any such three-spread, therefore, satisfies five such equations.

Again, consider a three-spread the equations of which can be put into one of the forms

$$x_i = f_i(u_1, u_2, u_3) = g_i(u_3) + u_1 g_i^3(u_3) + u_2 g_i^{33}(u_3), \qquad i = 0, 1, \ldots, n;$$
$$x_i = f_i(u_1, u_2, u_3) = g_i(u_3) + u_1 g_i^3(u_3) + u_2 k_i, \qquad i = 0, 1, \ldots, n;$$
$$x_i = f_i(u_1, u_2, u_3) = g_i(u_3) + u_1 l_i + u_2 k_i, \qquad i = 0, 1, \ldots, n,$$

in which k_i and l_i are constants.

In each case, the three-spread satisfies the equations:

$$f^{11} = 0, \qquad f^{12} = 0, \qquad f^{13} = 0,$$
$$f^{13} = af + bf^1 + cf^2 + df^3,$$
$$f^{23} = a_1 f + b_1 f^1 + c_1 f^2 + d_1 f^3.$$

Hence, *the necessary and sufficient condition that a three-spread satisfy five homogeneous linear partial differential equations of the second order is that it be either (a) an hypersurface in an S_4 or (b) generated by planes in such a way that consecutive planes intersect in a line.*

On Three-Spreads Which Satisfy Four Homogeneous Linear Partial Differential Equations of the Second Order.

12. Through an arbitrary point of a three-spread which satisfies four homogeneous linear partial differential equations of the second order, pass four lines having contact of the second order with the three-spread. For, let

$$\Omega_i = \sum_{h=1}^{3} \sum_{k=1}^{3} \omega_{hk} f_i^{hk}, \qquad \Psi_i = \sum_{h=1}^{3} \sum_{k=1}^{3} \psi_{hk} f_i^{hk}, \qquad i = 0, 1, \ldots, n.$$

By a suitable choice of ω_{hk} and ψ_{hk} the four given equations may be replaced by

$$f^{hk} = a_{hk} + b_{hk} f^1 + c_{hk} f^2 + d_{hk} f^3 + e_{hk} \Psi + g_{hk} \Omega, \qquad h, k = 1, 2, 3. \quad (1)$$

Hence, in the matrix which determines the three-point tangents (¶ 7), ϕ_i may be replaced by

$$\theta_i = \Psi_i \sum_{h=1}^{3} \sum_{k=1}^{3} e_{hk}\, du_h\, du_k + \Omega_i \sum_{h=1}^{3} \sum_{k=1}^{3} g_{hk}\, du_h\, du_k.$$

Hence, the lines through the point whose parameters are (u_1, u_2, u_3), for which $du_1 : du_2 : du_3$ satisfy

$$E = \sum_{h=1}^{3} \sum_{k=1}^{3} e_{hk}\, du_h\, du_k = 0 \quad \text{and} \quad G = \sum_{h=1}^{3} \sum_{k=1}^{3} g_{hk}\, du_h\, du_k = 0, \tag{2}$$

are three-point tangents to the three-spread. Solving equations (2) for the ratios $du_1 : du_2 : du_3$ and integrating, we find four systems of ∞^2 curves on the three-spread the tangents to which are three-point tangents to the three-spread. These curves correspond, on the three-spreads of the types under consideration, to the asymptotic lines on a surface in S_3.

13. By a transformation of the curvilinear coordinates (u_1, u_2, u_3), we may take one of the four systems of curves whose tangents are three-point tangents to the three-spread to be the curves $u_2 = \text{const.}$, $u_3 = \text{const.}$ Suppose this transformation effected. It then follows that

$$f^{11} = a_{11}f + b_{11}f^1 + c_{11}f^2 + d_{11}f^3. \tag{3}$$

Suppose, first, that the three-spread does *not* satisfy any equation of the form

$$af + bf^1 + cf^2 + df^3 + ef^{12} + gf^{13} = 0. \tag{4}$$

It then follows that the remaining three equations satisfied by the three-spread can be written in the form:

$$f^{22} = a_{22}f + b_{22}f^1 + c_{22}f^2 + d_{22}f^3 + e_{22}f^{12} + g_{22}f^{13}, \tag{5}$$

$$f^{23} = a_{23}f + b_{23}f^1 + c_{23}f^2 + d_{23}f^3 + e_{23}f^{12} + g_{23}f^{13}, \tag{6}$$

$$f^{33} = a_{33}f + b_{33}f^1 + c_{33}f^2 + d_{33}f^3 + e_{33}f^{12} + g_{33}f^{13}. \tag{7}$$

Differentiating equations (3), (5), (6) and (7) and substituting the values of f^{11}, f^{22}, f^{23} and f^{33} from these equations, it is seen that all the derivatives of order higher than the second can be expressed in terms of f, f^1, f^2, f^3, f^{12} and f^{13}.

Let (u_1, u_2, u_3) be given a fixed set of values which determine an arbitrary point on the three-spread. Since the three-spread does not satisfy an equation of the form (4), the six points

$$x_i = f_i, \quad x_i = f_i^1, \quad x_i = f_i^2, \quad x_i = f_i^3, \quad x_i = f_i^{12}, \quad x_i = f_i^{13}, \quad i = 0, 1, \ldots, n, \tag{8}$$

determine an S_5. The three-spread lies in this S_5. For, let $u_1 = \phi_1(t)$, $u_2 = \phi_2(t)$, $u_3 = \phi_3(t)$ determine an analytic curve C on the three-spread.

Since the successive derivatives of $f_i(t)$ with respect to t may be expressed linearly and homogeneously in terms of $f_i, f_i^1, f_i^2, f_i^3, f_i^{12}, f_i^{13}$, C has, at each point P on it, contact of higher than the fifth order with the S_5 (8) corresponding to P. Hence, these S_5 coincide and contain C. It follows that the three-spread lies in an S_5. For, if not, seven points taken at random on it would not lie in an S_5. But through any seven such points an analytic curve C can be passed. They therefore lie in an S_5.

If the three-spread does satisfy an equation of the type (4), it follows from equation (2) that $du_2 = du_3 = 0$ counts twice, at least, as a solution of $E = 0$ and $G = 0$. The tangents to $u_2 =$ const., $u_3 =$ const. therefore count twice as three-point tangents to the three-spread. Hence, *if through an arbitrary point of the three-spread passes a tangent to the three-spread whose direction is determined by a set of values of $du_1 : du_2 : du_3$ which count only once as solutions of equations* (2), *then the three-spread lies in an S_5.*

It will be shown hereafter that if, at an arbitrary point, every solution of (2) counts twice, at least, as a solution, then the three-spread may lie in a space of any number of dimensions.

14. Through each point of the three-spread pass, in general, in addition to the four curves determined by equations (2), three other curves of especial importance. To derive them, notice, first, that the section of the three-spread by an hyperplane through the tangent S_3 at an arbitrary point P is a two-spread having a node at P. The tangent quadric cone at P to this two-spread lies in the tangent S_3 to the three-spread and contains the four three-point tangents to the three-spread at P. If the hyperplane varies in such a way as always to contain the tangent S_3, this tangent cone therefore describes a pencil of cones through the three-point tangents. The cones of this pencil are generated by the lines through P determined by

$$\sum_{h=1}^{3} \sum_{k=1}^{3} e_{hk}\, du_h\, du_k + \lambda \sum_{h=1}^{3} \sum_{k=1}^{3} g_{hk}\, du_h\, du_k = 0,$$

where λ is a parameter the variation of which determines the different cones of the pencil.

Three cones of this pencil are composite. Their double lines pass through P. Hence, *through P pass three tangents to the three-spread which are double lines of tangent quadric cones at P to sections of the three-spread by hyperplanes through the tangent S_3 at P.* These three tangents will be referred to as the three "funda-

mental tangents" at P. To determine the direction of one of these tangents, it is necessary to determine a root λ of

$$| e_{hk} + \lambda g_{hk} | = 0,$$

and then determine the ratios $du_1 : du_2 : du_3$ from

$$\sum_{h=1}^{3} (e_{hk} + \lambda g_{hk}) du_h = 0, \qquad k = 1, 2, 3. \tag{9}$$

Integrating these three equations for each value of λ, we obtain three systems ∞^2 of curves on the three-spread whose tangents are fundamental tangents to the three-spread.

15. Let $(u_1 + du_1, u_2 + du_2, u_3 + du_3)$ be a point of the three-spread consecutive to P in any direction. *The tangent S_3 at this consecutive point intersects the tangent S_3 at P in a line through P the direction of which corresponds to the direction determined by $du_1 : du_2 : du_3$ in an involutorial birational quadratic transformation of which the coincidences are the three-point tangents at P and the fundamental lines are the three fundamental tangents at P.*

For, the tangent S_3 at the consecutive point is determined, to infinitesimals of the second order, by the four points:

$$\left.\begin{array}{ll} x_i = f_i + \sum\limits_{h=1}^{3} f_i^h \, du_h, & i = 0, 1, \ldots, n; \\[2ex] x_i = f_i^k + \sum\limits_{h=1}^{3} f_i^{hk} \, du_h, & k = 1, 2, 3; \quad i = 0, 1, \ldots, n. \end{array}\right\} \tag{10}$$

From equations (1) we deduce:

$$\left.\begin{array}{l} f_i^k + \sum\limits_{h=1}^{3} f_i^{hk} \, du_h = f_i^k + f_i \sum\limits_{h=1}^{3} a_{hk} \, du_h + f_i^1 \sum\limits_{h=1}^{3} b_{hk} \, du_h + f_i^2 \sum\limits_{h=1}^{3} c_{hk} \, du_h \\[2ex] \qquad + f_i^3 \sum\limits_{h=1}^{3} d_{hk} \, du_h + \Psi_i \sum\limits_{h=1}^{3} e_{hk} \, du_h + \Omega_i \sum\limits_{h=1}^{3} g_{hk} \, du_h, \\[2ex] \qquad\qquad\qquad k = 1, 2, 3; \quad i = 0, 1, \ldots, n. \end{array}\right\} \tag{11}$$

From (10) the equations of the tangent S_3 at the consecutive point may be written, in terms of the parameters (v_1, v_2, v_3), in the form:

$$x_i = f_i + \sum_{h=1}^{3} f_i^h \, du_h + \sum_{k=1}^{3} v_k \left(f_i^k + \sum_{h=1}^{3} f_i^{hk} \, du_h \right), \qquad i = 0, 1, \ldots, n.$$

Substituting into this expression the values of $f_i^k + \sum\limits_{h=1}^{3} f_i^{hk} \, du_h$ from (11), it is

seen that the points which lie in the tangent S_3 at the consecutive point and also in the tangent S_3 at P, *i. e.* in the S_3 determined by the four points

$$x_i = f_i, \quad x_i = f_i^1, \quad x_i = f_i^2, \quad x_i = f_i^3, \qquad i = 0, 1, \ldots, n,$$

are those which satisfy the equations:

$$\left. \begin{aligned} v_1 \sum_{h=1}^{3} e_{1h}\, du_h + v_2 \sum_{h=1}^{3} e_{2h}\, du_h + v_3 \sum_{h=1}^{3} e_{3h}\, du_h = 0, \\ v_1 \sum_{h=1}^{3} g_{1h}\, du_h + v_2 \sum_{h=1}^{3} g_{2h}\, du_h + v_3 \sum_{h=1}^{3} g_{3h}\, du_h = 0. \end{aligned} \right\} \quad (12)$$

Since these two equations are linear, the two S_3 intersect in a straight line. Since the equations are satisfied when $v_1 = v_2 = v_3 = 0$, the line goes through the point $(u_1 + du_1,\ u_2 + du_2,\ u_3 + du_3)$. In the limit, therefore, as the consecutive point approaches P in the direction $du_1 : du_2 : du_3$, the line of intersection of the two consecutive tangent S_3 approaches the line through P in the direction determined by the values of the ratios $\delta u_1 : \delta u_2 : \delta u_3$ which satisfy

$$\sum_{h=1}^{3} \sum_{k=1}^{3} e_{hk}\, du_h\, \delta u_k = 0, \qquad \sum_{h=1}^{3} \sum_{k=1}^{3} g_{hk}\, du_h\, \delta u_k = 0. \qquad (13)$$

But the line so determined is the intersection of the polars of the line $du_1 : du_2 : du_3$ with respect to the quadric cones determined by equations (2). The correspondence between the two directions at P is therefore an involutorial birational correspondence. The four united lines of the correspondence are the four intersections of the cones $E = 0$ and $G = 0$, *i. e.* the three-point tangents at P. The three fundamental lines of the correspondence are the three double lines of the pencil $E + \lambda\, G = 0$, *i. e.* the three fundamental tangents at P.

15. If, in particular, a consecutive point is taken in the direction of any one of the fundamental tangents, then the tangent S_3 at the consecutive point intersects the tangent S_3 at P in the plane joining the other two fundamental tangents. It therefore lies in an S_4 through the tangent S_3 at P. It is, in fact, easily seen that, if $du_1 : du_2 : du_3$ is a solution of equations (9), then the tangent S_3 at $(u_1 + du_1,\ u_2 + du_2,\ u_3 + du_3)$ lies, to infinitesimals of the second order, in the S_4 determined by

$$x_i = f_i,\ x_i = f_i^1,\ x_i = f_i^2,\ x_i = f_i^3,\ x_i = \Omega_i - \lambda\, \Psi_i, \qquad i = 0, 1, \ldots, n.$$

The tangent S_3 to the three-spread along an integral curve of (9), therefore, form, in general, the osculating S_3 to a curve C. Each osculating plane to C touches the three-spread at a point P of the integral curve of equations (9) and

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contains the other two fundamental tangents to the three-spread at the point of tangency. The tangent lines to C do not, in general, touch the three-spread. In fact, since the tangent lines to C are the lines of intersection of the tangent S_3 to the three-spread at three consecutive points of the integral curve of equations (9), if the tangent to C passed through P, the tangent S_3 at three consecutive points of the integral curve of (9) would pass through P. The tangent at P to the integral curve of (9) would therefore be a three-point tangent at P to the three-spread. This is not, in general, the case.

We shall now discuss the behavior of the three-spread when the tangent cones to the sections of the three-spread by the hyperplanes through the tangent S_3 at an arbitrary point P on the three-spread satisfy certain particular conditions. Consider, first, the case in which:

Case 1. *The Tangent Cones at P Are All Composite.*

A. Let the three-spread be generated by planes.

16. If a three-spread is generated by planes, an hyperplane through the tangent S_3 at an arbitrary point P on the three-spread intersects the three-spread in a two-spread the tangent cone to which, at P, has the generating plane through P for a component. It is, therefore, certainly composite.

A three-spread generated by planes satisfies, in general, however, only three homogeneous linear partial differential equations of the second order. In fact, if the equations of the three-spread are reduced to the form

$$x_i = f_i(u_1, u_2, u_3) = f_{i1}(u_3) + u_1 f_{i2}(u_3) + u_2 f_{i3}(u_3), \qquad i = 0, 1, \ldots, n,$$

we have identically only

$$f^{11} = 0, \qquad f^{12} = 0, \qquad f^{22} = 0. \tag{1}$$

Impose, now, the condition that the three-spread satisfy a fourth homogeneous linear partial differential equation of the second order:

$$a_{33} f^{33} + a_{23} f^{23} + a_{13} f^{13} + a_3 f^3 + a_2 f^2 + a_1 f^1 + af = 0. \tag{2}$$

Suppose, first, that $a_{33} \neq 0$. Then the tangent to the three-spread at an arbitrary point for which $du_1 : du_2 : du_3 = a_{13} : a_{23} : 2a_{33}$ counts only once as a three-point tangent. The three-spread therefore lies in an S_5.

Suppose, now, that $a_{33} = 0$. It then follows that consecutive planes of the three-spread lie in an S_4 and therefore intersect in a point. They therefore

either all pass through a fixed point or touch a fixed curve. In fact, equations (2) may, in this case, be written in the form

$$a f_{i1} + b f_{i2} + c f_{i3} + d f_{i1}^3 + e f_{i2}^3 + g f_{i3}^3 = 0, \qquad i = 0, 1, \ldots, n,$$

where a, b, etc., are independent of u_1 and u_2, and where d, e and g are not all zero, since a_{23} and a_{13} are not both zero. By a linear transformation of u_1 and u_2 this equation may be reduced to

$$f_{i3}^3 = \alpha_1 f_{i1} + \alpha_2 f_{i2} + \alpha_3 f_{i3}, \qquad i = 0, 1, \ldots, n.$$

If $\alpha_1 = \alpha_2 = 0$, this may still further be reduced to

$$f_{i3}^3 = 0, \qquad i = 0, 1, \ldots, n.$$

The planes, therefore, all pass through a point.

If α_1 and α_2 are not both zero, let $\alpha_2 \neq 0$. By a linear transformation of u_1 and u_2 the equation may be reduced to

$$f_{i3}^3 = f_{i2}, \qquad i = 0, 1, \ldots, n.$$

The equations of the three-spread now reduce to

$$x_i = f_{i1}(u_3) + u_1 f_{i3}^3(u_3) + u_2 f_{i3}(u_3), \qquad i = 0, 1, \ldots, n.$$

The generating planes, therefore, all touch the curve $x_i = f_{i3}(u_3)$. The three-spread does not, in general, lie in an S_5.

17. Conversely, if a three-spread generated by planes lies in an S_5, it satisfies four homogeneous linear partial differential equations of the second order, since, as we have seen, any three-spread lying in an S_5 satisfies four such equations. If a three-spread is generated by planes passing through a fixed point, its equations may be put into the form

$$x_i = f_i(u_3) + u_1 f_{i2}(u_3) + u_2 k_i, \qquad i = 0, 1, \ldots, n,$$

where the quantities k_i are constants. It then satisfies the four equations:

$$f^{11} = 0, \quad f^{12} = 0, \quad f^{22} = 0, \quad f^{23} = 0.$$

Finally, if the three-spread is generated by planes touching a fixed curve, its equations may be put into the form

$$x_i = f_{i1}(u_3) + u_1 f_{i3}^3(u_3) + u_2 f_{i3}(u_3), \qquad i = 0, 1, \ldots, n.$$

It then satisfies the equations:

$$f^{11} = f^{12} = f^{22} = f^{23} - f^1 = 0.$$

Hence, *the necessary and sufficient condition that a three-spread generated by planes satisfy four homogeneous linear partial differential equations of the second*

order is that it either (a) lie in an S_5, or (b) be generated by planes all passing through a fixed point, or (c) be generated by planes all touching a fixed curve.

18. For the three-spreads of case (a) for which the coefficient a_{33} of equation (2) does not vanish, the birational correspondence considered above between the tangents at an arbitrary point P reduces to an involutorial projectivity having the generating plane through P for united plane and the discrete three-point tangent for united tangent. It follows, in fact, from equations (1) and (2) that the tangent S_3 at a point consecutive to P in the direction $du_1 : du_2 : du_3$ intersects the tangent S_3 at P in a line through P in a direction determined by $\delta u_1 : \delta u_2 : \delta u_3$, where

$$\delta u_1 = a_{13}\, du_3 - a_{33}\, du_1, \quad \delta u_2 = a_{23}\, du_3 - a_{33}\, du_2, \quad \delta u_3 = a_{33}\, du_3.$$

For the three-spreads of cases (b) and (c) it is similarly seen, since $a_{33} = 0$, that the correspondence is degenerate. To every direction through P corresponds the line joining P to the point of intersection of the plane through P with the consecutive plane.

B. The three-spread is not generated by planes.

19. The tangent cone at an arbitrary point P to the section of the three-spread by an arbitrary hyperplane through the tangent S_3 at P has a double line, since, by hypothesis, it is composite. Moreover, since the three-spread is not generated by planes, through P there does not pass a continuum of straight lines lying on the three-spread. For, suppose that through P there could pass such a continuum of lines. Since the three-spread is not generated by planes, the continuum could not lie in a plane. Moreover, the continuum, being supposed to lie on the three-spread, would lie upon the tangent cone at P to the section of the three-spread by an arbitrary hyperplane through the tangent S_3 at P. This tangent cone would therefore be invariant for every such hyperplane. This is impossible, since the three-spread satisfies only four homogeneous linear partial differential equations of the second order.

The three-spread is therefore generated by straight lines such that the tangent S_3 along each straight line is invariant.

Conversely, if a three-spread is generated by straight lines in such a way that the tangent S_3 is invariant along each straight line, then the section of the three-spread by an hyperplane through the tangent S_3 at an arbitrary point P has as a double line the line through P along which the tangent S_3 is invariant. The tangent cone at P to the section is therefore composed of two planes through this double line. It is therefore composite.

Hence, the necessary and sufficient condition that the tangent cone at P to the two-spread of section of a three-spread not generated by planes, by an arbitrary hyperplane through the tangent S_3 at P, be composite, is that the three-spread be generated by straight lines in such a way that the tangent S_3 is invariant along each line.

20. Let

$$x_i = f_i(u_1, u_2, u_3) = g_i(u_2, u_3) + u_1 h_i(u_2, u_3), \qquad i = 0, 1, \ldots, n,$$

be such a three-spread, the tangent S_3 being invariant along the lines $u_2 = \text{const.}$, $u_3 = \text{const.}$ We then have:

$$f^{11} = 0, \tag{1}$$
$$f^{12} = af + bf^1 + cf^2 + df^3, \tag{2}$$
$$f^{13} = a_1 f + b_1 f^1 + c_1 f^2 + d_1 f^3. \tag{3}$$

Differentiating (2) with respect to u_3 and (3) with respect to u_2 and subtracting, we obtain:

$$c_1 f^{22} + (d_1 - c) f^{23} - d f^{33} = a_2 f + b_2 f^1 + c_2 f^2 + d_2 f^3. \tag{4}$$

Such a three-spread therefore satisfies four homogeneous linear partial differential equations of the second order, unless $c_1 = d_1 - c = d = 0$.

21. Suppose, however, that $c_1 = d_1 - c = d = 0$. It follows that all the coefficients of (4) vanish, since the three-spread can not satisfy an equation of the first order. If we differentiate (2) and (3) with respect to u_1, the resulting equations must be proportional to u_1. Hence, (2) and (3) may be written in the form

$$\alpha h^2 + \beta g^2 + \gamma h + \delta g = 0,$$
$$\alpha h^3 + \beta g^3 + \gamma_1 h + \delta_1 g = 0,$$

where α, β, γ, etc., are independent of u_1. Moreover, α and β are not both zero, since, otherwise, equations (2) and (3) would reduce to equations of the first order. By a transformation of curvilinear coordinates which is linear in u_1, these two equations may therefore be reduced to the form:

$$h^2 = \xi g + \eta h, \qquad h^3 = \xi_1 g + \eta_1 h. \tag{5}$$

Differentiating the first of these equations with respect to u_3, the second with respect to u_2, and subtracting, we obtain:

$$\xi g^3 - \xi_1 g^2 = \sigma g + \tau h.$$

Hence $\xi = \xi_1 = 0$. For, otherwise, the three-spread would satisfy an homogeneous linear partial differential equation of the first order. Equations (5),

therefore, reduce to $h^2 = \eta h$ and $h^3 = \eta_1 h$. Hence the $n + 1$ functions h_i are of the form $h_i = k_i \phi (u_2, u_3)$, where the quantities k_i are constants. Hence, by a slight transformation of curvilinear coordinates which is linear in u_1, the functions h_i may be reduced to constants. The three-spread is therefore a cone.

Let the quantities h_i be reduced to constants, and suppose that, in addition to equations (1), (2) and (3), there exists a fourth homogeneous linear partial differential equation of the second order which is satisfied by the three-spread. This equation reduces at once to

$$A g^{22} + B g^{23} + C g^{33} = a_2 g + b_2 g^2 + c_2 g^3 + d_2 h. \qquad (4')$$

The coefficients of this equation must be independent of u_1, since, otherwise, the three-spread would satisfy more than four such equations.

Let θ be a solution of the equation

$$A \theta^{22} + B \theta^{23} + C \theta^{33} = a_2 \theta + b_2 \theta^2 + c_2 \theta^3 + d_2.$$

If we replace u_1 by $u_1 - \theta$ in the equation of the three-spread, equation (4') reduces at once to

$$A g^{22} + B g^{23} + C g^{33} = a_2 g + b_2 g^2 + c_2 g^3. \qquad (4'')$$

Hence the three-spread is the projection from a fixed point of a two-spread which satisfies an homogeneous partial differential equation of the second order.

Conversely, consider a three-spread cone,

$$x_i = f_i (u_1, u_2, u_3) = g_i (u_2, u_3) + u_1 h_i, \qquad i = 0, 1, \ldots, n,$$

which projects from a fixed point $x_i = h_i$ a two-spread which satisfies an equation of the type (4''). It is seen at once that the three-spread satisfies the four equations:

$$f^{11} = 0, \qquad f^{12} = 0, \qquad f^{13} = 0,$$
$$A f^{22} + B f^{23} + C f^{33} = a_2 f + b_2 f^2 + c_2 f^3 - a_2 u_1 f^1. \qquad (4'')$$

We have therefore proved that: *A three-spread generated by straight lines in such a way that the tangent S_3 along each line of the system is invariant satisfies four homogeneous linear partial differential equations of the second order, unless it is a cone. If the three-spread is a cone, it satisfies four such equations if, and only if, it is the projection from a fixed point of a two-spread satisfying an homogeneous linear partial differential equation of the second order.* In neither case does it lie, in general, in an S_5.

21. On a three-spread which satisfies equations (1), (2), (3) and either (4) or (4'''), the birational correspondence between the tangents at an arbitrary

point P is composite. To any direction through P corresponds the direction of the invariant line through P. For, if we write (4) or $(4''')$ in the form

$$\alpha_{22} f^{22} + \alpha_{23} f^{23} + \alpha_{33} f^{33} = \alpha f + \alpha_1 f^1 + \alpha_2 f^2 + \alpha_3 f^3,$$

and if we choose β_{22}, β_{23}, β_{33}, γ_{22}, γ_{23} and γ_{33} so as to make the determinant

$$| \alpha_{22} \ \beta_{23} \ \gamma_{33} |$$

different from zero, we find, for the direction through P corresponding to $du_1 : du_2 : du_3$, the intersection of the polars of (du_1, du_2, du_3) with respect to

$$(\alpha_{33}\beta_{23} - \alpha_{23}\beta_{33}) du_2^2 + 2(\alpha_{22}\beta_{33} - \alpha_{33}\beta_{22}) du_2 du_3 + (\alpha_{23}\beta_{22} - \alpha_{22}\beta_{23}) du_3^2 = 0,$$

$$(\alpha_{33}\gamma_{23} - \alpha_{23}\gamma_{33}) du_2^2 + 2(\alpha_{22}\gamma_{33} - \alpha_{33}\gamma_{22}) du_2 du_3 + (\alpha_{23}\beta_{33} - \alpha_{22}\beta_{23}) du_3^2 = 0.$$

The direction required is therefore determined by

$$\left.\begin{array}{l} [(\alpha_{33}\beta_{23} - \alpha_{23}\beta_{33}) du_2 + (\alpha_{22}\beta_{33} - \alpha_{33}\beta_{22}) du_3] \delta u_2 + [(\alpha_{22}\beta_{33} - \alpha_{33}\beta_{22}) du_2 + (\alpha_{23}\beta_{22} - \alpha_{22}\beta_{23}) du_3] \delta u_3 = 0, \\ [(\alpha_{33}\gamma_{23} - \alpha_{23}\gamma_{33}) du_2 + (\alpha_{22}\gamma_{33} - \alpha_{33}\gamma_{22}) du_3] \delta u_2 + [(\alpha_{22}\gamma_{33} - \alpha_{33}\gamma_{22}) du_2 + (\alpha_{23}\gamma_{22} - \alpha_{22}\gamma_{23}) du_3] \delta u_3 = 0. \end{array}\right\} (6)$$

Hence, in general, $\delta u_2 = \delta u_3 = 0$.

If, however, du_2 and du_3 satisfy the condition

$$\alpha_{22} du_3^2 - \alpha_{23} du_2 du_3 + \alpha_{33} du_2^2 = 0, \qquad (7)$$

equations (6) become identical. The tangent S_3 at the consecutive point $(u_1 + du_1, u_2 + du_2, u_3 + du_3)$ now intersects the tangent S_3 at P in a plane through the invariant line $du_2 = du_3 = 0$ and through the tangent to the two-spread $u_1 = $ const. determined by (6).

Moreover, if $du_2 : du_3$ is one solution of (7), then the value of $\delta u_2 : \delta u_3$ corresponding to it in equation (6) is the other solution of (7). Hence, *if P is an arbitrary point of the three-spread, two distinct or coincident planes pass through the invariant line through P such that, if P' is consecutive to P in one of these planes, the tangent S_3 at P' intersects the tangent S_3 at P in the other plane.*

22. *These planes are invariant as P moves along the invariant line.* For, differentiating equation (4) or $(4''')$ with respect to u_1, we find, since the three-spread satisfies only four homogeneous linear partial differential equations of the second order, that the ratios $\alpha_{22} : \alpha_{23} : \alpha_{33}$ are independent of u_1. Hence, equation (7) is independent of u_1.

If the three-spread is a cone, the above theorem follows at once from known properties of two-spreads satisfying an homogeneous linear partial differential equation of the second order.[*]

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[*] Segre, *loc. cit.*, p. 576.

23. We shall now distinguish two cases according to the equality or inequality of the roots of equation (7). Suppose first that the roots are unequal.

If the three-spread is a cone, it is, in this case, simply the projection of a two-spread having two distinct systems of "characteristic lines." *

If the three-spread is not a cone, it is generated in two ways by a system ∞^1 of cones or of developables whose generators are the invariant lines of the three-spread and whose edges of regression form a system of characteristic lines on a two-spread lying on the given three-spread.

For, since equation (7) is independent of u_1, and since the roots of (7) are unequal, they may, by a transformation of the parameters u_2 and u_3, be reduced to $du_2 = 0$ and $du_3 = 0$. We then have, in equation (4), $c_1 = d = 0$. Hence, from equations (1) and (2) it follows that the two-spreads $u_3 = $ const. are developables or cones. Similarly, from (1) and (3) it follows that the two-spreads $u_2 = $ const. are developables or cones.

If both systems are cones, the equations of the three-spread may be reduced to

$$x_i = f_i(u_1, u_2, u_3) = g_i(u_2) + u_1 h_i(u_3), \qquad i = 0, 1, \ldots, n.$$

The three-spread is therefore generated by the lines cutting each of two curves.

Conversely, any three-spread generated by the common secants to two curves, has its tangent S_3 invariant along the secant lines. For, putting the equations of the three-spread in the above form, we obtain:

$$f^{11} = f^{12} = f^{23} = f^3 - u_1 f^{13} = 0.$$

If the two-spreads of at least one of the two systems are developables, the equations of the three-spread may be put into the form:

$$x_i = f_i(u_1, u_2, u_3) = g_i(u_2, u_3) + u_1 g_i^3(u_2, u_3), \qquad i = 0, 1, \ldots, n.$$

From equation (2) it follows that

$$g^{23} = \alpha g + \beta g^2 + \gamma g^3. \tag{8}$$

Hence, the two-spread,

$$x_i = g_i(u_2, u_3), \qquad i = 0, 1, \ldots, n,$$

formed by the edges of regression of the developables $u_2 = $ const., has two distinct systems of characteristic curves. From the form of equation (8) it is seen that one of these systems of characteristic curves is the system $u_2 = $ const.,

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† The "*caratieristiche*" of Segre.

i. e. the edges of regression themselves. The other system is formed by the curves $u_3 = $ const. These are the curves in which the generators of the second system of developables or cones touch the two-spread. Hence, *the generators of the three-spread which touch the two-spread along a characteristic curve of the second system generate a developable or cone.*

Conversely, let

$$x_i = g_i (u_2, u_3), \qquad i = 0, 1, \ldots, n,$$

be a two-spread having two distinct systems of characteristic curves, and let $u_2 = $ const. and $u_3 = $ const. determine these two systems of curves so that

$$g^{23} = \alpha g + \beta g^2 + \gamma g^3.$$

Then the three-spread generated by the tangents to either system of characteristic curves, for example, the three-spread

$$x_i = f_i (u_1, u_2, u_3) = g_i (u_2, u_3) + u_1 g_i^3 (u_2, u_3), \qquad i = 0, 1, \ldots, n,$$

satisfies the four equations

$$f^{11} = 0,$$
$$f^{12} = af + bf^1 + cf^2 + df^3,$$
$$f^{13} = a_1 f + b_1 f^1 + c_1 f^2 + d_1 f^3,$$
$$f^{23} = a_2 f + b_2 f^1 + c_2 f^2 + d_2 f^3,$$

and therefore has the lines $u_2 = $ const., $u_3 = $ const. for invariant lines. From the form of the last equation, the roots of equation (7) are seen to be distinct.

Hence, *the necessary and sufficient condition that a three-spread be generated by lines in such a way that the tangent S_3 along each line is invariant and that the two planes of intersecton of this S_3 with the tangent S_3 along consecutive generators are distinct is that the three-spread (a) be the projection from a fixed point of a two-spread having two distinct systems of characteristic curves, or (b) be generated by the lines intersecting each of two fixed curves, or (c) be generated by the tangents to a system of characteristic curves on a two-spread having two distinct systems of such curves.*

24. Suppose, next, that the roots of (7) are equal. If this condition is satisfied, the cones which intersect in the three-point tangents at P have a common component. The plane so determined is a component of the tangent cone at P to the section of the three-spread by every hyperplane through the tangent S_3 at P. Every line in the plane through P is, therefore, a three-point tangent at P. Since this plane is invariant as P moves along the invariant line, every line in the plane has three-point contact with the three-spread at its intersection with the invariant line.

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Conversely, if a three-spread which does not satisfy more than four homogeneous linear partial differential equations of the second order has, at each point, a continuum of three-point tangents, these must form a pencil, since, otherwise, the section by an arbitrary hyperplane would lie in an S_3. The plane of this pencil must be a component of the tangent cone at its vertex to the section of the three-spread by any hyperplane through the tangent S_3 at the vertex. Hence, the three-spread, since it is not generated by planes, is generated by lines in such a way that the tangent S_3 is invariant along each line, and also in such a way that the roots of equation (7) are equal.

If the three-spread is a cone, the condition that the roots of (7) be equal reduces, when the values of α_{22}, α_{23} and α_{33} are substituted in from equation (4'''), to

$$B^2 - 4AC = 0.$$

This, however, is the condition that the two-spread $x_i = g_i(v_2, u_3)$ have, at each point, a three-point tangent. A three-spread cone, therefore, will have, at each point, a pencil of three-point tangents if, and only if, it is the projection of a two-spread having, at each point, a three-point tangent.

If the three-spread is not a cone, the condition that the roots of (7), *i. e.* of (4), be equal reduces to

$$(d_1 - c)^2 + 4c_1 d = 0.$$

Since the ratios $c_1 : d_1 - c : - d$ are independent of u_1, equation (4) may be reduced to the form:

$$f^{22} = a_2 f + b_2 f^1 + c_2 f^2 + d_2 f^3.$$

We now have $d = 0$, $d_1 = c$. From equations (1) and (2), therefore, it follows that the two-spreads $u_3 = $ const. are developables or cones. They can not be cones; for, if they were, the three-spread itself would be conical, or else generated by planes. They are therefore developables. The equations of the three-spread may, therefore, be put into the form

$$x_i = f_i(u_1, u_2, u_3) = g_i(u_2, u_3) + u_1 g_i^2(u_2, u_3), \qquad i = 0, 1, \ldots, n.$$

From equations (2) and (3) we now obtain

$$g^{22} = \alpha g + \beta g^2 + \gamma g^3.$$

Hence the edges of regression of the developables $u_3 = $ const. are three-point tangent curves on the two-spread:

$$x_i = g_i(u_2, u_3), \qquad i = 0, 1, \ldots, n.$$

Conversely, consider a three-spread generated by a system of three-point tangents to a two-spread. Let the equations of the three-spread be

$$x_i = f_i(u_1, u_2, u_3) = g_i(u_2, u_3) + u_1 g_i^2(u_2, u_3), \qquad i = 0, 1, \ldots, n,$$

in which, since the generators of the three-spread are three-point tangents to the two-spread,

$$g^{22} = \alpha g + \beta g^2 + \gamma g^3.$$

It therefore follows that the three-spread satisfies the four equations:

$$f^{11} = 0,$$

$$f^{12} = \frac{1}{u_1} f^2 - \frac{1}{u_1} f^1,$$

$$f^{13} = a_1 f + b_1 f^1 + c_1 f^2 + \frac{1}{u_1} f^3,$$

$$f^{22} = a_2 f + b_2 f^1 + c_2 f^2 + d_2 f^3.$$

Hence, *the necessary and sufficient condition that a three-spread not generated by planes nor lying in an S_4 have, at an arbitrary point, a pencil of three-point tangents is that it (a) be a cone projecting a two-spread which has, at each point, a three-point tangent, or (b) be generated by a system ∞^2 of three-point tangents to a two-spread.*

It will be supposed throughout, hereafter, that the tangent cones at P to the sections of the three-spread by the hyperplanes through the tangent S_3 at P are not all composite. Of the four generators of intersection of the cones of this pencil at P two or more may be consecutive. It will next be determined under what conditions it may happen that:

Case 2. Two Three-Point Tangents Are Consecutive.

25. Let $u_2 = $ const., $u_3 = $ const. determine the system of curves the tangents to which count twice as three-point tangents to the three-spread. Since the cones $\sum_{h,k=1}^{3} e_{hk} du_h du_k = 0$ and $\sum_{h,k=1}^{3} g_{hk} du_h du_k = 0$ touch along $du_2 = 0$, $du_3 = 0$, two of the four differential equations satisfied by the three-spread may be written in the form:

$$f^{11} = a_1 f + b_1 f^1 + c_1 f^2 + d_1 f^3, \tag{1}$$

$$f^{13} = a_2 f + b_2 f^1 + c_2 f^2 + d_2 f^3 + e_2 f^{12}. \tag{2}$$

It will be supposed, in this case, that the other two three-point tangents at an arbitrary point are distinct. The three-spread therefore lies in an S_5.

Most of the results of the present discussion of this case, however, hold for any three-spread which satisfies two equations which can be reduced to the form of equations (1) and (2), and which therefore contains a system ∞^2 of curves the tangents to which count twice as three-point tangents.

Since the cones $\sum\limits_{h=1}^{3}\sum\limits_{k=1}^{3} e_{hk}\,du_h\,du_k = 0$ and $\sum\limits_{h=1}^{3}\sum\limits_{k=1}^{3} g_{hk}\,du_h\,du_k = 0$, at an arbitrary point P on the three-spread, touch along $du_2 = du_3 = 0$, the tangent determined by $du_2 = du_3 = 0$ also counts twice as a fundamental tangent in the birational correspondence (¶ 15) between the tangents at P. Hence, the tangent S_3 at a consecutive point $(u_1 + du_1,\ u_2,\ u_3)$ on $u_2 = \text{const.}$, $u_3 = \text{const.}$ intersects the tangent S_3 at $(u_1,\ u_2,\ u_3)$ in a plane which passes through $du_2 = du_3 = 0$ and the discrete fundamental line.

26. Suppose, first, that the plane of intersection of consecutive tangent S_3 along an arbitrary curve of the system $u_2 = \text{const.}$, $u_3 = \text{const.}$ is invariant. The curve $u_2 = \text{const.}$, $u_3 = \text{const.}$ lies in this plane and the three-spread is touched along the curve by the plane.

The three-spreads which are touched along non-rectilinear plane curves by a system ∞^2 of planes will be discussed in Case 5. Suppose therefore, for the present case, that the curves $u_2 = \text{const.}$, $u_3 = \text{const.}$ are straight lines. *The necessary and sufficient condition that a ruled three-spread be touched along each generator by a fixed plane is that the three-spread be generated by a system ∞^1 of developables or cones.*

That the condition is sufficient follows from the fact that the tangent plane to the developable or cone touches the three-spread along the generator. To show that the condition is necessary, let the equations of the three-spread be written in the form:

$$x_i = f_i(u_1,\ u_2,\ u_3) = g_i(u_2,\ u_3) + u_1 h_i(u_2,\ u_3), \qquad i = 0, 1, \ldots, n.$$

The tangent S_3 at $(u_1,\ u_2,\ u_3)$ is determined by the generator through the point and by the line L joining $x_i = g_i^2 + u_1 h_i$ to $x_i = g_i^3 + u_1 h_i^3$. Since, for all values of u_1, the tangent S_3 contains a fixed plane, the line L must, for all values of u_1, meet the plane in a fixed point or in a fixed line intersecting the generator through $(u_1,\ u_2,\ u_3)$. In either case, there exists a value of u_1 such that

$$a(g^2 + u_1 h^2) + b(g^3 + u_1 h^3) + cg + dh = 0.$$

By a suitable transformation of the parameters this equation may be reduced

either to $g^3 = h$ or else to $g^3 = 0$. Hence, the three-spread is generated either by developables or by cones.

27. Suppose, next, that the tangent S_3 along $u_2 = \text{const.}$, $u_3 = \text{const.}$ all contain a fixed straight line. Since three consecutive S_3 of the system all pass through each point of the curve $u_2 = \text{const.}$, $u_3 = \text{const.}$, the straight line common to the tangent S_3 must coincide with $u_2 = \text{const.}$, $u_3 = \text{const.}$ The three-spread is therefore ruled. Let its equations be :

$$x_i = f_i(u_1, u_2, u_3) = g_i(u_2, u_3) + u_1 h_i(u_2, u_3), \qquad i = 0, 1, \ldots, n.$$

Equation (2) is now the condition that the six points

$$x_i = g_i, \quad x_i = h_i, \quad x_i = g_i^2, \quad x_i = h_i^2, \quad x_i = g_i^3, \quad x_i = h_i^3, \quad i = 0, 1, \ldots, n, \quad \text{(a)}$$

lie in an S_4. Conversely, if these six points lie in an S_4, equation (2) is satisfied and the generators count twice as three-point tangents. But the six points (a) lie in S_4 if, and only if, the tangent planes to the two-spreads $x_i = g_i(u_2, u_3)$ and $x_i = h_i(u_2, u_3)$, at corresponding points, intersect in a point. Hence, *the necessary and sufficient condition that the rectilinear generators of a ruled three-spread count twice as three-point tangents to the three-spread is that they join corresponding points of two two-spreads the tangent planes to which, at corresponding points, intersect in a point.*

The tangent S_3 to the three-spread along the generator $u_2 = \text{const.}$, $u_3 = \text{const.}$ envelope a quadric hypersurface in the S_4 determined by the points (a). This hypersurface has the generator for double line. For, the tangent S_3 at (u_1, u_2, u_3) is determined by the generator through it and the line joining

$$x_i = g_i^2 + u_1 h_i^2 \text{ to } x_i = g_i^3 + u_1 h_i^3, \qquad i = 0, 1, \ldots, n.$$

When u_1 varies, the latter line describes a regulus and the S_3 determined by it and the fixed generator envelopes a quadric hypersurface.

28. Suppose, finally, that the tangent S_3 along an arbitrary curve of the system $u_2 = \text{const.}$, $u_3 = \text{const.}$ have, at most, a point in common. We have seen that the tangent S_3 at $(u_1 + du_1, u_2, u_3)$ intersects the tangent S_3 at (u_1, u_2, u_3) in a plane through $du_2 = du_3 = 0$ and the discrete fundamental line. Three consecutive tangent S_3 along $u_2 = \text{const.}$, $u_3 = \text{const.}$, therefore, intersect in a line. Since the osculating plane to $u_2 = \text{const.}$, $u_3 = \text{const.}$ lies in the

tangent S_3, this line goes through (u_1, u_2, u_3). It coincides, in fact, with the discrete fundamental line at (u_1, u_2, u_3). For, let

$$f^{22} = a_{22}f + b_{22}f^1 + c_{22}f^2 + d_{22}f^3 + e_{22}f^{12} + g_{22}f^{33}, \qquad (3)$$

$$f^{23} = a_{23}f + b_{23}f^1 + c_{23}f^2 + d_{23}f^3 + e_{23}f^{12} + g_{23}f^{33} \qquad (4)$$

be the remaining two differential equations satisfied by the three-spread. Then the line of intersection of three consecutive tangent S_3 passes through (u_1, u_2, u_3) in the direction

$$\frac{e_2 \cdot e_2 \cdot g_{22} - 2 e_2 g_{23} + 1}{e_2 \cdot e_2 \cdot (e_{23} g_{22} - e_{22} g_{23}) + e_2 e_{22} - e_{23}} \, du_1 = -\frac{du_2}{e_2} = du_3.$$

But this is also the direction of the discrete fundamental line at (u_1, u_2, u_3). Along each curve $u_2 = \text{const.}$, $u_3 = \text{const.}$, therefore, the discrete fundamental lines generate a developable or cone. This developable or cone, obviously, touches the three-spread along $u_2 = \text{const.}$, $u_3 = \text{const.}$ If the locus of the discrete fundamental lines is a developable, the osculating S_3 to the edge of regression to any point is the tangent S_3 to the three-spread at the corresponding point.

Conversely, let $u_2 = \text{const.}$, $u_3 = \text{const.}$ be a system of curves on a three-spread

$$x_i = f_i(u_1, u_2, u_3), \qquad i = 0, 1, \ldots, n,$$

such that the tangent S_3 along an arbitrary curve of the system envelope a developable which touches the three-spread along the corresponding curve $u_2 = \text{const.}$, $u_3 = \text{const.}$ Let the equations of the three-spread generated by the edges of regression of these developables be

$$x_i = F_i(u_1, u_2, u_3), \qquad i = 0, 1, \ldots, n.$$

It then follows that:

$$f_i = \alpha F_i + \beta F_i^1,$$
$$f_i^1 = \alpha_1 F_i + \beta_1 F_i^1 + \gamma_1 F_i^{11}, \qquad f_i^2 = \alpha_2 F_i + \beta_2 F_i^1 + \gamma_2 F_i^{11} + \delta_2 F_i^{111}, \qquad i = 0, 1, \ldots, n,$$
$$f_i^3 = \alpha_3 F_i + \beta_3 F_i^1 + \gamma_3 F_i^{11} + \delta_3 F_i^{111}, \qquad i = 0, 1, \ldots, n.$$

From these equations, it follows that the three-spread $x_i = f_i$ satisfies equations (1) and (2). It is similarly seen that if the tangent S_3 to the given three-spread along $u_2 = \text{const.}$, $u_3 = \text{const.}$ envelope a cone which touches the three-spread along $u_2 = \text{const.}$, $u_3 = \text{const.}$, then equations (1) and (2) are satisfied. Hence, *the necessary and sufficient condition that a system ∞^2 of twisted curves which generate a three-spread count twice as three-point tangent curves, is that the tangent S_3 to the three-spread along each curve of the system envelope a developable or cone which touches the three-spread along the curve corresponding to it.*

Case 3. *Two Pairs of Three-Point Tangents Are Consecutive.*

29. Let the directions of the two three-point tangents at an arbitrary point be $du_2 = du_3 = 0$ and $du_1 = 0$, $du_3 = a\,du_2$. Denote $\dfrac{\partial}{\partial u_2} + a\dfrac{\partial}{\partial u_3}$ by Δ. Since the three-spread has only a finite number of three-point tangents, Δf^1 does not lie in the tangent S_3. The four differential equations may therefore be reduced to:

$$f^{11} = a_1 f + b_1 f^1 + c_1 \Delta f + d_1 f^3, \tag{1}$$

$$f^{13} = a_2 f + b_2 f^1 + c_2 \Delta f + d_2 f^3 + e_2 \Delta f^1, \tag{2}$$

$$\Delta(\Delta f) = a_3 f + b_3 f^1 + c_3 \Delta f + d_3 f^3, \tag{3}$$

$$\Delta f^3 = a_4 f + b_4 f^1 + c_4 \Delta f + d_4 f^3 + e_4 \Delta f^1. \tag{4}$$

From (1), the point $x_i = \Delta f^{11}$ is seen to lie in the S_4 determined by the five points:

$$x_i = f_i, \quad x_i = f_i^1, \quad x_i = f_i^2, \quad x_i = f_i^3, \quad x_i = \Delta f_i^1, \quad i = 0, 1, \ldots, n. \tag{5}$$

Hence the point $x_i = \dfrac{\partial}{\partial u_1}(\Delta f_i^1) = \Delta f_i^{11} + \dfrac{\partial a}{\partial u_1} f_i^{13}$ lies in this S_4. Similarly, it is seen from (3) that the point $x_i = \Delta(\Delta f_i^1)$ lies in the same S_4. Differentiating (2) with respect to u_1, $x_i = f_i^{113}$ is seen to lie in the S_4 of (5). Hence, differentiating (1) with respect to u_3, we find $d_1 = 0$, since $x_i = f_i^{33}$ can not lie in this S_4. In the corresponding way it can be shown that $d_3 = $ C. From (2) we find that $x_i = \Delta f^{13}$ lies in the S_4 of (5). Similarly, from (4), $x_i = \dfrac{\partial}{\partial u_1}(\Delta f^3)$ lies in this S_4.

But $\dfrac{\partial}{\partial u_1}(\Delta f^3) = \Delta f^{13} + \dfrac{\partial a}{\partial u_1} f^{33}$. Hence, $\dfrac{\partial a}{\partial u_1} = 0$, since $x_i = f_i^{33}$ does not lie in the S_4 of (5). By a transformation of curvilinear coordinates of the form $u_1 = \bar{u}_1$, $u_2 = \phi(\bar{u}_2, \bar{u}_3)$, $u_3 = \psi(\bar{u}_2, \bar{u}_3)$, therefore, a may be reduced to zero.

Suppose this transformation effected. The four equations now become:

$$f^{11} = a_1 f + b_1 f^1 + c_1 f^2, \tag{1'}$$

$$f^{13} = a_2 f + b_2 f^1 + c_2 f^2 + d_2 f^3 + e_2 f^{12}, \tag{2'}$$

$$f^{22} = a_3 f + b_3 f^1 + c_3 f^2, \tag{3'}$$

$$f^{23} = a_4 f + b_4 f^1 + c_4 f^2 + d_4 f^3 + e_4 f^{12}. \tag{4'}$$

From the first and third equations it follows that the two-spread $u_3 = $ const. has two distinct systems of three-point tangents at each point on it. It therefore lies in an S_3. This S_3 is determined by

$$x_i = f_i, \quad x_i = f_i^1, \quad x_i = f_i^2, \quad x_i = f_i^{12}, \quad i = 0, 1, \ldots, n.$$

The consecutive S_3 is determined, to infinitesimals of the second order, by

$$x_i = f_i + f_i^3 du_3, \quad x_i = f_i^1 + f_i^{13} du_3, \quad x_i = f_i^2 + f_i^{23} du_3, \quad x_i = f_i^{12} + f_i^{123} du_3; \quad i = 0, 1, \ldots, n.$$

But $x_i = f_i^{23}$ and $x_i = f_i^{123}$ lie in the S_4 of

$$x_i = f_i, \quad x_i = f_i^1, \quad x_i = f_i^2, \quad x_i = f_i^3, \quad x_i = f_i^{12}, \quad i + 0, 1, \ldots, n.$$

Hence, the S_3 which contain any two consecutive two-spreads $u_3 = $ const. lie in an S_4 and therefore intersect in a plane.

The equations of the three-spread can therefore always be put in one of the following forms:

$$x_i = f_i(u_1, u_2, u_3) = l_0(u_1, u_2, u_3) g_i(u_3) + l_1(u_1, u_2, u_3) g_i^3(u_3)$$
$$+ l_2(u_1, u_2, u_3) g_i^{33}(u_3) + l_3(u_1, u_2, u_3) g_i^{333}(u_3), \quad i = 0, 1, \ldots, n; \quad (6)$$

$$x_i = f_i(u_1, u_2, u_3) = l_0(u_1, u_2, u_3) g_i(u_3) + l_1(u_1, u_2, u_3) g_i^3(u_3)$$
$$+ l_2(u_1, u_2, u_3) g_i^{33}(u_3) + l_3(u_1, u_2, u_3) M_i, \quad i = 0, 1, \ldots, n; \quad (7)$$

$$x_i = f_i(u_1, u_2, u_3) = l_0(u_1, u_2, u_3) g_i(u_3) + l_1(u_1, u_2, u_3) g_i^3(u_3)$$
$$+ l_2(u_1, u_2, u_3) L_i + l_3(u_1, u_2, u_3) M_i, \quad i = 0, 1, \ldots, n; \quad (8)$$

$$x_i = f_i(u_1, u_2, u_3) = l_0(u_1, u_2, u_3) g_i(u_3) + l_1(u_1, u_2, u_3) K_i$$
$$+ l_2(u_1, u_2, u_3) L_i + l_3(u_1, u_2, u_3) M_i, \quad i = 0, 1, \ldots, n, \quad (9)$$

where K_i, L_i and M_i are constants.

30. Conversely, any three-spread, the equations of which are of any one of these four forms, and on which the asymptotic lines of the three-dimensional surfaces $u_3 = $ const. of the system

$$x_j = l_j(u_1, u_2, u_3), \quad j = 0, 1, 2, 3,$$

are distinct, has, at an arbitrary point, two tangents each of which counts twice as a three-point tangent.

For, transform the parameters u_1 and u_2 in such a way that $u_2 = $ const. and $u_3 = $ const. are the asymptotic lines of the surfaces $u_3 = $ const. Then

$$l_j^{11} = a_1 l_j + b_1 l_j^1 + c_1 l_j^2, \quad j = 0, 1, 2, 3;$$
$$l_j^{22} = a_2 l_j + b_2 l_j^1 + c_2 l_j^2, \quad j = 0, 1, 2, 3.$$

From these equations, equations (1') and (3') follow.

Moreover, $x_i = f_i^{13}$ and $x_i = f_i^{23}$ lie in the S_4 determined by the tangent S_3 and $x_i = f_i^{12}$, from which equations (2') and (4') follow.

Hence, *the necessary and sufficient condition that a three-spread have, at an arbitrary point, two pairs of consecutive three-point tangents is that it be generated by a system ∞^1 of surfaces, the asymptotic lines of which are distinct and which lie in a system ∞^1 of S_3 such that consecutive S_3 intersect in a plane.* Such a three-spread does not, in general, lie in an S_5.

Case 4. Three Three-Point Tangents Are Consecutive.

31. Suppose, first, that the threefold three-point tangent curves are straight lines. Let the equations of the three-spread be:

$$x_i = f_i(u_1, u_2, u_3) = g_i(u_2, u_3) + u_1 h_i(u_2, u_3), \qquad i = 0, 1, \ldots, n.$$

The conditions that the lines $u_2 = \text{const.}$, $u_3 = \text{const.}$ count three times as three-point tangent lines are that the three-spread satisfy the three equations:

$$f^{11} = 0, \tag{1}$$

$$f^{13} = a_2 f + b_2 f^1 + c_2 f^2 + d_2 f^3 + e_2 f^{12}, \tag{2}$$

$$f^{33} - 2 e_2 f^{23} + e_2 \cdot e_2 f^{22} = a_3 f + b_3 f^1 + c_3 f^2 + d_3 f^3 + e_3 f^{12}. \tag{3}$$

By differentiating (2) with respect to u_1, we find that $d_2 e_2 = -\left(c_2 + \dfrac{\partial e_2}{\partial u_1}\right)$. But, by differentiating (2) with respect to u_2 and u_3 and (3) with respect to u_1, we find that $d_2 e_2 = 2\dfrac{\partial e_2}{\partial u_1} - c_2$. Hence $\dfrac{\partial e_2}{\partial u_1} = 0$. Hence, e_2 is independent of u_1. By a transformation of the parameters u_2 and u_3 which is independent of u_1, therefore, e_2 may be reduced to zero. Suppose this done. Since c_2 is now also zero, the two-spreads $u_2 = \text{const.}$ are either developables or cones.

32. First, let the two-spreads $u_2 = \text{const.}$ be developables. The equations of the three-spread may now be written in the form:

$$x_i = f_i(u_1, u_2, u_3) = g_i(u_2, u_3) + u_1 g_i^3(u_2, u_3), \qquad i = 0, 1, \ldots, n.$$

Equations (1) and (2) are now satisfied. The S_4 of

$$x_i = f_i, \quad x_i = f_i^1, \quad x_i = f_i^2, \quad x_i = f_i^3, \quad x_i = f_i^{12}, \qquad i = 0, 1, \ldots, n,$$

coincides, for all values of u_1, with the S_4 of

$$x_i = g_i, \quad x_i = g_i^2, \quad x_i = g_i^3, \quad x_i = g_i^{23}, \quad x_i = g_i^{33}, \qquad i = 0, 1, \ldots, n.$$

Since $e_2 = 0$, (3) reduces to the condition that

$$g^{333} = a_4 g + b_4 g^2 + c_4 g^3 + d_4 g^{23} + e_4 g^{33}. \tag{4}$$

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Hence, *the necessary and sufficient condition that a three-spread have, at an arbitrary point, two pairs of consecutive three-point tangents is that it be generated by a system ∞^1 of surfaces, the asymptotic lines of which are distinct and which lie in a system ∞^1 of S_3 such that consecutive S_3 intersect in a plane.* Such a three-spread does not, in general, lie in an S_5.

Case 4. Three Three-Point Tangents Are Consecutive.

31. Suppose, first, that the threefold three-point tangent curves are straight lines. Let the equations of the three-spread be:

$$x_i = f_i(u_1, u_2, u_3) = g_i(u_2, u_3) + u_1 h_i(u_2, u_3), \qquad i = 0, 1, \ldots, n.$$

The conditions that the lines $u_2 = \text{const.}$, $u_3 = \text{const.}$ count three times as three-point tangent lines are that the three-spread satisfy the three equations:

$$f^{11} = 0, \tag{1}$$

$$f^{13} = a_2 f + b_2 f^1 + c_2 f^2 + d_2 f^3 + e_2 f^{12}, \tag{2}$$

$$f^{33} - 2 e_2 f^{23} + e_2 \cdot e_2 f^{22} = a_3 f + b_3 f^1 + c_3 f^2 + d_3 f^3 + e_3 f^{12}. \tag{3}$$

By differentiating (2) with respect to u_1, we find that $d_2 e_2 = -\left(c_2 + \dfrac{\partial e_2}{\partial u_1}\right)$. But, by differentiating (2) with respect to u_2 and u_3 and (3) with respect to u_1, we find that $d_2 e_2 = 2\dfrac{\partial e_2}{\partial u_1} - c_2$. Hence $\dfrac{\partial e_2}{\partial u_1} = 0$. Hence, e_2 is independent of u_1. By a transformation of the parameters u_2 and u_3 which is independent of u_1, therefore, e_2 may be reduced to zero. Suppose this done. Since c_2 is now also zero, the two-spreads $u_2 = \text{const.}$ are either developables or cones.

32. First, let the two-spreads $u_2 = \text{const.}$ be developables. The equations of the three-spread may now be written in the form:

$$x_i = f_i(u_1, u_2, u_3) = g_i(u_2, u_3) + u_1 g_i^3(u_2, u_3), \qquad i = 0, 1, \ldots, n.$$

Equations (1) and (2) are now satisfied. The S_4 of

$$x_i = f_i, \quad x_i = f_i^1, \quad x_i = f_i^2, \quad x_i = f_i^3, \quad x_i = f_i^{12}, \qquad i = 0, 1, \ldots, n,$$

coincides, for all values of u_1, with the S_4 of

$$x_i = g_i, \quad x_i = g_i^2, \quad x_i = g_i^3, \quad x_i = g_i^{23}, \quad x_i = g_i^{33}, \qquad i = 0, 1, \ldots, n.$$

Since $e_2 = 0$, (3) reduces to the condition that

$$g^{333} = a_4 g + b_4 g^2 + c_4 g^3 + d_4 g^{23} + e_4 g^{33}. \tag{4}$$

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Hence, at any point P of any curve $u_2 = $ const. on the two-spread $x_i = g_i(u_2, u_3)$, the osculating S_3 to the curve lies in the S_4 determined by the tangent planes to $x_i = g_i(u_2, u_3)$ at P and at the consecutive point on $u_2 = $ const.

Conversely, let

$$x_i = g_i(u_2, u_3), \qquad i = 0, 1, \ldots, n,$$

be a two-spread in S_n which contains a system ∞^1 of curves such that, at an arbitrary point P of a curve of the system, the osculating S_3 to the curve lies in the S_4 determined by the tangent planes to the two-spread at P and at the consecutive point along the curve. Such a two-spread satisfies a differential equation which can be reduced to the form (4). The tangents to the curves of the system generate a three-spread the equations of which then become:

$$x_i = f_i(u_1, u_2, u_3) = g_i(u_2, u_3) + u_1 g_i^3(u_2, u_3), \qquad i = 0, 1, \ldots, n.$$

This three-spread satisfies equations (1), (2) and (3), and therefore its generators count thrice as three-point tangents.

33. Suppose, now, that the two-spreads $u_2 = $ const. are cones. Let the equations of the three-spread be written in the form:

$$x_i = f_i(u_1, u_2, u_3) = g_i(u_2, u_3) + u_1 h_i(u_2), \qquad i = 0, 1, \ldots, n.$$

Since $e_2 = 0$, the condition that (3) be satisfied reduces to

$$g^{33} = a_5 g + b_5 g^2 + c_5 g^3 + d_5 h + e_5 h^2. \qquad (5)$$

Hence, at any point P of an arbitrary curve $u_2 = $ const. on $x_i = g_i(u_2, u_3)$, the osculating plane to the curve lies in the S_4 determined by the tangent plane to $x_i = g_i(u_2, u_3)$ at P and the tangent line to $x_i = h_i(u_2)$ at the vertex of the cone through P.

Conversely, let V be a two-spread and C a curve in S_n. Let there correspond to the points P_1 of C the curves C_1 of a system on V such that the osculating plane to an arbitrary curve C_1 of the system at any point P on it lies in the S_4 determined by the tangent plane to V at P_1 and the tangent line to C at P_1. Then the parametric equations $x_i = g_i(u_2, u_3)$ of V and $x_i = h_i(u_2)$ of C can be set up in such a way that (5) is satisfied for all values of i, and therefore that the three-spread

$$x_i = g_i(u_2, u_3) + u_1 h_i(u_2), \qquad i = 0, 1, \ldots, n,$$

generated by the lines joining the points of each curve C_1 to the corresponding point P of C, satisfies (1), (2) and (3), and therefore has its generators for three-fold three-point tangents.

Hence e_2 is independent of u_1, and, by a transformation of the parameters u_2 and u_3, may be reduced to zero. The four equations now become:

$$f^{11} = 0, \tag{1'}$$

$$f^{13} = a_2 f + b_2 f^1 + c_2 f^2 + d_2 f^3, \tag{2'}$$

$$f^{33} = a_3 f + b_3 f^1 + c_3 f^2 + d_3 f^3 + e_3 f^{13}, \tag{3'}$$

$$f^{23} = a_4 f + b_4 f^1 + c_4 f^2 + d_4 f^3 + e_4 f^{12}. \tag{4'}$$

Differentiating (2') with respect to u_2 and (4') with respect to u_1, we find $c_2 = 0$. Hence, the two-spreads $u_2 = \text{const.}$ are developables or cones. Differentiating (1'), (2') and (3') successively with respect to u_1 and u_3, we find that the developable or cone $u_2 = \text{const.}$ lies in the S_4 determined by

$$x_i = f_i, \quad x_i = f_i^1, \quad x_i = f_i^2, \quad x_i = f_i^3, \quad x_i = f_i^{12}, \qquad i = 0, 1, \ldots, n.$$

But $x_i = f_i^2$, and its consecutive derivative points with respect to u_1 and u_3, also lie in this S_4. Hence, the consecutive two-spread, corresponding to the value $u_2 + du_2$ of the parameter, also lies in this S_4. Each two-spread, therefore, lies in the intersection of two consecutive S_4. Hence the three-spread is generated by a system ∞^1 of developables or cones lying in a system ∞^1 of S_3 such that consecutive S_3 lie in an S_4 and therefore intersect in a plane.

Conversely, it is seen, by an argument analogous to that given in Case 3, that a three-spread generated in this manner satisfies four equations of the type (1), (2), (3), (4) and, therefore, that its rectilinear generators count four times as three-point tangents.

Hence, *the necessary and sufficient condition that the generators of a ruled three-spread count four times as three-point tangent curves is that the three-spread be generated by a system ∞^1 of developables or cones lying in a system of S_3 such that consecutive S_3 intersect in a plane.* The three-spread does not, in general, lie in an S_5.

- 37. Suppose, now, that the fourfold three-point tangent curves $u_2 = \text{const.}$, $u_3 = \text{const.}$ are not straight lines. The conditions that the curves $u_2 = \text{const.}$, $u_3 = \text{const.}$ count four times as three-point tangents are:

$$f^{11} = a_1 f + b_1 f^1 + c_1 f^2 + d_1 f^3, \tag{1}$$

$$f^{13} - e_2 f^{12} = a_2 f + b_2 f^1 + c_2 f^2 + d_2 f^3, \tag{2}$$

$$f^{33} - e_2 f^{23} = a_3 f + b_3 f^1 + c_3 f^2 + d_3 f^3 + e_3 f^{12}, \tag{3}$$

$$f^{23} - e_2 f^{22} = a_4 f + b_4 f^1 + c_4 f^2 + d_4 f^3 + e_4 f^{12}. \tag{4}$$

From (3) and (4) we obtain at once:

$$f^{33} - 2 e_2 f^{23} + e_2 e_2 f^{22} = a_5 f + b_5 f^1 + c_5 f^2 + d_5 f^3 + e_5 f^{12}. \tag{5}$$

Moreover, $e_5 \neq 0$, since $du_1 = 0$, $du_2 + e_2 du_3 = 0$ does not determine a three-point tangent.

Differentiating (2) with respect to u_2 and u_3, we find that

$$x_i = f_i^{133} - 2 e_2 f^{123} + e_2 e_2 f^{122}$$

lies in the S_4 of

$$x_i = f_i, \quad x_i = f_i^1, \quad x_i = f_i^2, \quad x_i = f_i^3, \quad x_i = f_i^{12}, \quad i = 0, 1, \ldots, n. \tag{6}$$

Hence, by differentiating (5) with respect to u_1, we find that $x_i = f_i^{112}$ lies in this S_4. Differentiating (1) with respect to u_1, we now find that $c_1 + d_1 e_2 = 0$. Hence (1) may be written in the form:

$$f^{11} = a_6 f + b_6 f^1 + d_6 (f^3 - e_2 f^2). \tag{1'}$$

Differentiating this equation successively with respect to u_1, it is found that the curve $u_2 = \text{const.}$, $u_3 = \text{const.}$ lies in the S_4 of (6). It follows that this curve must be a plane curve. For, otherwise, by expressing f^2 and f^3 in terms of f, f^1, f^{11} and f^{111}, it would be seen that the entire three-spread lay in this S_4. Hence:

$$f^{111} = a_6 f + b_6 f^1 + c_6 f^{11}. \tag{7}$$

From equations (1) and (7) all the others follow. Hence, *if a system of non-rectilinear plane curves are three-point tangent curves to the three-spread, they are fourfold three-point tangent curves.*

38. Let the equations of the three-spread be put in the form

$$x_i = \phi_0 (u_1, u_2, u_3) \, g_{0i} (u_2, u_3) + \phi_1 (u_1, u_2, u_3) \, g_{1i} (u_2, u_3) + \phi_2 (u_1, u_2, u_3) \, g_{2i} (u_2, u_3),$$
$$i = 0, 1, \ldots, n.$$

By a linear transformation of ϕ_0, ϕ_1, ϕ_2, the two-spreads $x_i = g_{0i}$, $x_i = g_{1i}$ and $x_i = g_{2i}$ may be taken to be two-spreads which lie on the three-spread. Suppose this transformation effected. For each pair of values of u_2 and u_3, the curve $u_2 = \text{const.}$, $u_3 = \text{const.}$ goes through the points $x_i = g_{0i}$, $x_i = g_{1i}$ and $x_i = g_{2i}$.

Equation (1') now becomes:

$$\phi_0 (g_0^3 - e_2 g_0^2) + \phi_1 (g_1^3 - e_2 g_1^2) + \phi_2 (g_2^3 - e_2 g_2^2) = \alpha \, g_0 + \beta \, g_1 + \gamma \, g_2. \tag{1''}$$

Let e_{20}, e_{21} and e_{22} be the values of e_2 at $x_i = g_{0i}$, $x_i = g_{1i}$ and $x_i = g_{2i}$ respectively. Substituting into (1'') the values of u_1 which give rise to these three points and simplifying (1'') by means of the resulting equations, we obtain:

$$\phi_0 (e_{20} - e_2) g_0^2 + \phi_1 (e_{21} - e_2) g_1^2 + \phi_2 (e_{22} - e_2) g_2^2 = \alpha_1 g_0 + \beta_1 g_1 + \gamma_1 g_2.$$

But the plane of $x_i = g_{0i}^2$, $x_i = g_{1i}^2$, $x_i = g_{2i}^2$ can not have more than one point in the plane of $x_i = g_{0i}$, $x_i = g_{1i}$, $x_i = g_{2i}$. For, otherwise, the five points (6) would lie in an S_3 and the three-spread would satisfy five homogeneous linear partial differential equations of the second order. Hence,

$$\frac{\phi_0 (e_{20} - e_2)}{\sigma_0} = \frac{\phi_1 (e_{21} - e_2)}{\sigma_1} = \frac{\phi_2 (e_{22} - e_2)}{\sigma_2},$$

where σ_0, σ_1 and σ_2 are independent of u_1.

Eliminating e_2, we now obtain:

$$\phi_0 \phi_1 \sigma_2 (e_{20} - e_{21}) + \phi_1 \phi_2 \sigma_0 (e_{21} - e_{22}) + \phi_2 \phi_0 \sigma_1 (e_{22} - e_{20}) = 0.$$

Since this equation is of second degree in ϕ_0, ϕ_1 and ϕ_2, the curves $u_2 = \text{const.}$, $u_3 = \text{const.}$ are conics.

Consider the four-spread generated by the planes of these conics,

$$x_i = g_{0i} (u_2, u_3) + v_1 g_{1i} (u_2, u_3) + v_2 g_{2i} (u_2, u_3), \qquad i = 0, 1, \dots, n,$$

where v_1, v_2, u_2, u_3 are the parameters. The tangent S_4 at any point is the S_4 of the points (6). It is therefore invariant over each plane $u_2 = \text{const.}$, $u_3 = \text{const.}$

39. Conversely, let

$$x_i = g_{0i} (u_2, u_3) + v_1 g_{1i} (u_2, u_3) + v_2 g_{2i} (u_2, u_3), \qquad i = 0, 1, \dots, n, \qquad (8)$$

be a four-spread generated by planes in such a way that the tangent S_4 is invariant along each plane. Let $x_i = \theta_i$ and $x_i = \psi_i$ be two of the six points,

$$x_i = g_{0i}^2, \ x_i = g_{1i}^2, \ x_i = g_{2i}^2, \ x_i = g_{0i}^3, \ x_i = g_{1i}^3, \ x_i = g_{2i}^3, \qquad i = 0, 1, \dots, n,$$

which do not lie in the plane $u_2 = \text{const.}$, $u_3 = \text{const.}$ Let

$$g_0^2 = \alpha_1 g_0 + \beta_1 g_1 + \gamma_1 g_2 + \sigma_0 \theta + \tau_0 \psi,$$
$$g_0^3 = \alpha_2 g_0 + \beta_1 g_1 + \gamma_1 g_2 + \pi_0 \theta + \rho_0 \psi,$$

and similarly for g_1^2, g_1^3, g_2^2 and g_2^3.

Then the points in the plane $u_2 = \text{const.}$, $u_3 = \text{const.}$ which also lie in a consecutive plane must satisfy the equations:

$$du_2 (\sigma_0 + v_1 \sigma_1 + v_2 \sigma_2) + du_3 (\pi_0 + v_1 \pi_1 + v_2 \pi_2) = 0,$$
$$du_2 (\tau_0 + v_1 \tau_1 + v_2 \tau_2) + du_3 (\rho_0 + v_1 \rho_1 + v_2 \rho_2) = 0.$$

They therefore lie on the conic:

$$\begin{vmatrix} \sigma_0 + v_1 \sigma_1 + v_2 \sigma_2 & \pi_0 + v_1 \pi_1 + v_2 \pi_2 \\ \tau_0 + v_1 \tau_1 + v_1 \tau_2 & \rho_0 + v_1 \rho_1 + v_2 \rho_2 \end{vmatrix} = 0. \qquad (9)$$

This conic may be composite. Its locus is then a ruled three-spread. Excluding this case, let

$$v_1 = \frac{\phi_1(u_1, u_2, u_3)}{\phi_0(u_1, u_2, u_3)}, \qquad v_2 = \frac{\phi_2(u_1, u_2, u_3)}{\phi_0(u_1, u_2, u_3)}$$

satisfy (9) identically. Then the three-spread,

$$x_i = f_i(u_1, u_2, u_3) = \phi_0(u_1, u_2, u_3) g_{0i}(u_2, u_3) + \phi_1(u_1, u_2, u_3) g_{1i}(u_2, u_3)$$
$$+ \phi_2(u_1, u_2, u_3) g_{2i}(u_1, u_2), \qquad i = 0, 1, \ldots, n, \qquad (10)$$

satisfies two equations of the forms (1') and (7). The conics $u_2 = $ const., $u_3 = $ const. on it therefore count four times as three-point tangent curves.

Since the conics (9) are not composite, the four-spread (8) satisfies nine homogeneous linear partial differential equations of the second order. Since the four-spread is not generated by S_3 in such a way that consecutive S_3 intersect in a plane, it lies in an S_5. Hence, the three-spread (10) lies in an S_5.

Hence, *the necessary and sufficient condition that a system ∞^2 of non-rectilinear plane curves which generate a three-spread be three-point tangent curves to the three-spread, is that they be the system of conics in which each plane of a four-spread which is generated by planes in such a way that the tangent S_4 is invariant along each plane is intersected by the planes consecutive to it. The conics then count four times as three-point tangent curves to the three-spread. The three-spread generated by such a system of conics lies in an S_5.*

The three-spread is, in this case, touched along the conics by the planes of the conics. When a three-spread generated by a system ∞^2 of plane curves is touched along the curves by the planes of the curves, the tangents to the curves are three-point tangents to the three-spread. Hence:

The necessary and sufficient condition that a three-spread be touched along non-rectilinear curves by a system ∞^2 of planes is that the curves of contact be conics and that the plane of each conic meet the planes consecutive to it in the points of the conic.

Some Properties of Lines in Space of Four Dimensions and their Interpretation in the Geometry of the Circle in Space of Three Dimensions.

By C. L. E. Moore.

The study of systems of lines in space of four dimensions is so closely related to the study of systems of circles and other geometric configurations that it has seemed worth while to develop this geometry and to interpret the results in the geometry of the circle. Klein[*] has suggested the study of lines in space of four dimensions as a possible approach to the study of systems of circles. The only extensive study made of systems of lines in S_4 is that of Castelnuovo;[†] but this is concerned only with linear systems of lines, and as linear systems of circles have already been studied by Koenigs[‡] and Cosserat,[§] no further reference will be made to linear systems of lines. The properties of lines treated in this paper involve the first derivatives.

Ruled Surfaces.

We will define a line in a space of four dimensions S_4 by means of the four equations

$$
\left.
\begin{aligned}
x_1 &= a_1 + \alpha_1 z, \\
x_2 &= a_2 + \alpha_2 z, \\
x_3 &= a_3 + \alpha_3 z, \\
x_4 &= z;
\end{aligned}
\right\} \quad (1)
$$

and to define a ruled surface we will consider $a_1, a_2, a_3, \alpha_1, \alpha_2, \alpha_3$ as functions of a variable t. Throughout this paper the functions are supposed to be continuous and to possess first derivatives which are also finite and continuous.

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[*] "Einleitung in die Höhere Geometrie," Vol. I, p. 242.

[†] *Atti del reale Instituto Veneto*, 1890–91.

[‡] "Contributions à la théorie du cercle dans l'espace," *Annales de la Faculté des Sciences de Toulous*, t. II, 1888.

[§] "Sur la cercle considéré comme élément générateur de l'espace," *Annales de Toulous*, t. III, 1889.

The tangent plane at a point z, t of the ruled surface (1) (the coefficients being considered functions of t) is then

$$x_1 = a_1 + \alpha_1 z + \tau \alpha_1 + t_1 (a_1' + z \alpha_1'),$$
$$x_2 = a_2 + \alpha_2 z + \tau \alpha_2 + t_1 (a_2' + z \alpha_2'),$$
$$x_3 = a_3 + \alpha_3 z + \tau \alpha_3 + t_1 (a_3' + z \alpha_3'),$$
$$x_4 = z + \tau,$$

where t_1, τ are the parameters and $a', \alpha', \ldots$ denote $\dfrac{da}{dt}, \dfrac{d\alpha}{dt}, \ldots$. If we express this plane as the intersection of two hyperplanes, we have

$$\begin{vmatrix} x_1 - a_1 - \alpha_1 z & x_2 - a_2 - \alpha_2 z & x_4 - z \\ \alpha_1 & \alpha_2 & 1 \\ a_1' + \alpha_1' z & a_2' + \alpha_2' z & 0 \end{vmatrix} = 0,$$

$$\begin{vmatrix} x_1 - a_1 - \alpha_1 z & x_3 - a_3 - \alpha_3 z & x_4 - z \\ \alpha_1 & \alpha_3 & 1 \\ a_1' + \alpha_1' z & a_3' + \alpha_3' z & 0 \end{vmatrix} = 0.$$

Developing these determinants, we have

$$\frac{x_1 - a_1 - \alpha_1 x_4}{x_2 - a_2 - \alpha_2 x_4} = \frac{a_1' + \alpha_1' z}{a_2' + \alpha_2' z} = \rho_1, \qquad \qquad (2)$$

$$\frac{x_1 - a_1 - \alpha_1 x_4}{x_3 - a_3 - \alpha_3 x_4} = \frac{a_1' + \alpha_1' z}{a_3' + \alpha_3' z} = \rho_2. \qquad \qquad (3)$$

The first member of each of these equations is free from z, and therefore we see each of these hyperplanes depends linearly on z. Now given z, and the two hyperplanes, and consequently the tangent plane, is determined; and inversely, given ρ_1, then z is determined and consequently ρ_2. Hence, *there is a* $(1, 1)$ *correspondence between the points of a generator r of a ruled surface and the planes tangent to the surface in these points;* that is, the Chasles correlation holds for ruled surfaces in S_4. *

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* From (2) and (3) we have, equating the values of z obtained from each,

$$(a_2 a_3 - a_3 a_2) \rho_1 \rho_2 + (a_1 a_2 - a_2 a_1) \rho_1 - (a_1 a_3 - a_3 a_1) \rho_2 = 0 \qquad (4)$$

as the relation which must hold between ρ_1, ρ_2. If this relation holds, the locus of the plane [(2), (3)] is

$$(a_2 a_3 - a_3 a_2) \frac{x_1 - a_1 - a_1 x_4}{x_2 - a_2 - a_2 x_4} \cdot \frac{x_1 - a_1 - a_1 x_4}{x_3 - a_3 - a_3 x_4} + (a_1 a_2 - a_2 c_1) \frac{x_1 - a_1 - a_1 x_4}{x_2 - a_2' - a_2 x_4} - (a_1 a_3 - a_3 a_1) \frac{x_1 - a_1 - a_1 x_4}{x_3 - a_3 - a_3 x_4} = 0,$$

which is the equation of the hyperplane generated by the tangent planes to the ruled surface in points of a generator r.

We may consider a_1', a_2', a_3', α_1', α_2', α_3' as the homogeneous coordinates of this projectivity (correlation). If

$$a_1' \alpha_2' - a_2' \alpha_1' = 0, \tag{5}$$

$$a_1' \alpha_3' - a_3' \alpha_1' = 0, \tag{6}$$

the hyperplanes (2) and (3) are the same for every value of z; that is, for every point of the generator r. Hence the surface has the same tangent plane the whole length of the generator r. Equations (5) and (6) are then the conditions in order that the correlation be degenerate. When this condition is given we easily obtain the tangent plane from equations (2) and (3). We will call such a plane a *focal plane* or *singular plane*.

If we solve (2) and (3) for z, we have

$$z = - \frac{a_1' - a_2' \rho_1}{\alpha_1' - \alpha_2' \rho_1}, \tag{2'}$$

$$z = - \frac{a_1' - a_3' \rho_2}{\alpha_1' - \alpha_3' \rho_2} \tag{3'}$$

(remembering that ρ_1 and ρ_2 are not independent). We see that (5) and (6) are also the conditions in order that z be independent of ρ_1 and ρ_2. In this case to the point z of r corresponds each plane which passes through r. Such a point we will call a *focus* or *singular point*. It is easily seen also that (5) and (6) are the conditions in order that the generator infinitely close to r intersect r, for the line infinitely close to r has for equations

$$x_1 = a_1 + \alpha_1 z + d a_1 + z d \alpha_1,$$
$$x_2 = a_2 + \alpha_2 z + d a_2 + z d \alpha_2,$$
$$x_3 = a_3 + \alpha_3 z + d a_3 + z d \alpha_3,$$
$$x_4 = z.$$

Subtracting equations (1), we have

$$d a_1 + z d \alpha_1 = 0,$$
$$d a_2 + z d \alpha_2 = 0,$$
$$d a_3 + z d \alpha_3 = 0,$$

and (5) and (6) are at once seen to be the conditions in order that these equations be consistent.

Now if we consider a_1', a_2', a_3', α_1', α_2', α_3' as the homogeneous coordinates of a space Σ of five dimensions, each point of Σ will represent a correlation between the points of r and a singly infinite number of planes passing through r

and generating an ordinary space S_3. In this space Σ the intersection of the two cones whose equations are (5) and (6) represent the degenerate correlations. The intersection of these two cones is a variety V_3^3 of three dimensions and of the third order; since the S_3 $a_1' = \alpha_1' = 0$ is common to both cones, it is evident that the points of this S_3 do not represent degenerate correlations. We will designate this variety V_3^3 by Φ.

From equations (5) and (6) we see that the planes defined by

$$\left. \begin{array}{l} a_1' + z\alpha_1' = 0, \\ a_2' + z\alpha_2' = 0, \\ a_3' + z\alpha_3' = 0 \end{array} \right\} \quad (7)$$

stand in the variety Φ for all values of z and represent the points of the generator r; that is, the points of r considered as *foci* are represented by (7). We also see that the lines represented by

$$\left. \begin{array}{l} a_1' - \rho_1 a_2' = 0, \\ \alpha_1' - \rho_1 \alpha_2' = 0, \\ a_1' - \alpha_3' \rho_2 = 0, \\ \alpha_1' - \alpha_3' \rho_2 = 0 \end{array} \right\} \quad (8)$$

stand in Φ for all values of ρ_1 and ρ_2 and represent the planes passing through r; that is, the planes passing through r and considered as focal planes are represented by the lines (8). The lines of (8) which represent planes tangent to the the same ruled surface are given by the values of ρ_1 and ρ_2 which satisfy (4). Evidently each line represented by (8) will cut each plane represented by (7).

Congruences of Lines in S_4, C_2.

If a_1, a_2, a_3, α_1, α_2, α_3 are functions of two parameters u_1, u_2, equations (1) will generate a congruence of lines in S_4. Then the correlations which belong to a line r of the congruence, that is the correlations on r defined by all the ruled surfaces of the congruence which have r for generator, are represented in Σ by

$$(l) \left\{ \begin{array}{l} a_1' = \dfrac{\partial a_1}{\partial u_1}\dfrac{du_1}{dt} + \dfrac{\partial a_1}{\partial u_2}\dfrac{du_2}{dt}, \\[2mm] a_2' = \dfrac{\partial a_2}{\partial u_1}\dfrac{du_1}{dt} + \dfrac{\partial a_2}{\partial u_2}\dfrac{du_2}{dt}, \\[2mm] \cdots\cdots\cdots\cdots\cdots\cdots\cdots, \\[1mm] \cdots\cdots\cdots\cdots\cdots\cdots\cdots, \\[2mm] \alpha_3 = \dfrac{\partial \alpha_3}{\partial u_1}\dfrac{du_1}{dt} + \dfrac{\partial \alpha_3}{\partial u_2}\dfrac{du_2}{dt}. \end{array} \right.$$

[We have here supposed that u_1 and u_2 are both functions of a parameter t; then by varying these functions we get all the surfaces of the congruence which contain the line r.] The above equations show that the correlations in a congruence C_2 which belong to a line r of the congruence are defined in Σ by a line (l). $\left(\text{The parameters of the line are } \dfrac{du_1}{dt}, \dfrac{du_2}{dt}.\right)$ A line in Σ will not, in general, cut the variety Φ. Hence: *In general, there are no developables belonging to a congruence of lines in a space of four dimensions S_4; that is, there are no degenerate projectivities.*

Particular Cases. 1. The line (l) may intersect Φ in one point. In this case there is one focus on r and one focal plane passing through r (since the point in which (l) cuts Φ must lie on a plane of (7) and a line of (8)). If this condition is fulfilled for each line of the congruence, *the focal points generate a focal surface, but the focal planes in general do not have an envelope.* The congruence is formed by the tangents to a system of lines traced on a surface.

2. The line (l) may cut Φ in two points, in which case there are two foci on r and two focal planes passing through r. If this condition is fulfilled for every generator r, *the congruence will have two focal surfaces, and through each line will pass two developables belonging to the congruence.* Let M, N be the two foci on any generator r, and let μ, ν be the corresponding focal planes. The plane ν is tangent to the locus of M. For any line of the congruence is tangent to both developables, and therefore tangent to both the locus of M and the locus of N. Now as r moves, generating one of the developables whose cuspidal edge is on the locus of N, it is always tangent to the surface generated by M; hence, the tangent planes to this developable are tangent to the locus of M, but these tangent planes are the planes ν. *This congruence is formed by the lines tangent to a system of characteristics* traced on a surface in S_4.* Every surface in S_4 can be the focal surface for such a congruence.

3. The two points in which the line (l) cuts Φ may coincide, in which case the two focal surfaces coincide and *the congruence is generated by the tangent lines to a system of characteristics traced on a surface of the parabolic type.*†

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* See Segre: "Su una classe di superficii degli iperspazii, ecc.," *Atti di Torino*, 1907. In this paper Segre has discussed a class of surfaces in S on which can be traced two systems of curves such that the tangent planes to the surface, in points of one of these curves, form a developable. Such curves he has called characteristics. It is shown that every surface in S_4 belongs to this class.

† A surface is said to be of the parabolic type when the two systems of characteristics coincide. In this case the coordinates of a point of the surface satisfy a partial differential equation of the second order and of the parabolic type.

Specializations of the Focal Surface. In general the foci in cases 1, 2, 3 above generate surfaces, but they may generate curves in particular cases. In case 1, if the focal surface becomes a curve, the congruence will consist of the cones of lines whose vertices lie on the curve; but here again these cones have no envelope.

If one of the focal surfaces in case 2 is a curve, the second surface can not be a general surface but must be the envelope of ∞^1 cones whose vertices lie on the curve. This requires that the one system of characteristics should be such that the tangent planes to the surface along points of one of these characteristics should form a cone, and in general this will not be the case. The first system of focal planes consists of the tangent planes of the cones, and the second system consists of the planes passing through the lines of the congruence and tangent to the curve. The focal points on any line are the points where it cuts the curve and is tangent to the surface.

If both focal surfaces become curves, then the developables become the cones of lines whose vertices are on one curve and which touch the other curve. If the two curves are straight lines, the congruence becomes a congruence in S_3.

The two curves may coincide, and, as in S_3, the congruence consists of the pencils of lines whose planes are tangent to the curve and whose vertices lie on the curve.

Other specializations may occur, but in general they occur only for particular lines of the congruence and not identically. It may happen that the line (l) lies wholly within the variety Φ. This may occur in two ways: (a) it may lie in one of the planes (7), in which case there is one focus corresponding to this plane, but there are ∞^1 focal planes corresponding to the lines of (8) which are cut by the line (l); that is, *there is one focus on the line* r, *but* ∞^1 *planes passing through* r *are focal planes*. These focal planes generate an ordinary space S_3; that is, they form a pencil with r for axis. For if we consider any hyperplane in Σ, it will cut any given plane of (7) in a line s; and it is easily seen that the relation which holds between ρ_1 and ρ_2 for the lines of (8) which cut s is a bilinear relation without a constant term, and this by equations (2) and (3) determines a pencil of planes. (b) The line (l) may coincide with a line of (8), *in which case each point of* r *is a focus but only one plane is a focal plane.*

Examples of lines of the kind (a) would be the lines tangent to the curve of intersection of the two focal surfaces (if such exist). In this case we see that evidently the two foci have coincided in the point of tangency, but the two planes passing through r and tangent to the two focal surfaces are the two focal planes;

but we saw that if the foci coincide, the focal planes must coincide unless the line (l) lies in Φ. Hence ∞^1 planes of the pencil determined by these two focal planes are focal planes. Lines of the class (b) would be illustrated by the lines of the developable (if such exist) which envelopes the two sheets of the focal surface. Evidently the two points of contact of such a line r with the two sheets of the focal surface are foci; but the two focal planes have coincided, and as the focal planes can not coincide if the focal points do not coincide, unless the line (l) lies in Φ, and as there are two foci, there must be an infinite number.

Triply Infinite Systems of Lines. C_3.

If the coefficients $a_1, \ldots, a_1, \ldots$ are functions of three variables u_1, u_2, u_3, the coordinates of the correlation become

$$(\Pi) \quad \begin{cases} a_1' = \dfrac{\partial a_1}{\partial u_1}\dfrac{du_1}{dt} + \dfrac{\partial a_1}{\partial u_2}\dfrac{du_2}{dt} + \dfrac{\partial a_1}{\partial u_3}\dfrac{du_3}{dt}, \\[2mm] a_2' = \dfrac{\partial a_2}{\partial u_1}\dfrac{du_1}{dt} + \dfrac{\partial a_2}{\partial u_2}\dfrac{du_2}{dt} + \dfrac{\partial a_2}{\partial u_3}\dfrac{du_3}{dt}, \\[2mm] \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots, \\[1mm] \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots, \\[2mm] \alpha_3' = \dfrac{\partial \alpha_3}{\partial u_1}\dfrac{du_1}{dt} + \dfrac{\partial \alpha_3}{\partial u_2}\dfrac{du_2}{dt} + \dfrac{\partial \alpha_3}{\partial u_3}\dfrac{du_3}{dt}. \end{cases}$$

Hence the correlations which belong to any line r of the complex are represented in Σ by a plane (Π).

The plane Π will cut the variety Φ in three points; hence, *through each line of the triply infinite system pass three developables.*[*] The system of three parameters then consists of three systems of ∞^2 developables. The focal points generate three focal varieties of three dimensions $\phi_3', \phi_3'', \phi_3'''$. These varieties may be considered as the locus of the edges of regression of the three systems of developables. If we consider one of the developables whose edge of regression is on ϕ_3', for example, all the lines of the developable are tangent to ϕ_3'', ϕ_3''', and therefore the planes of these developables are tangent to ϕ_3'' and ϕ_3'''. Therefore, *if $M_1\mu_1$, $M_2\mu_2$, $M_3 u_3$ are the foci and corresponding focal planes of a line r, the plane μ_1, for example, is tangent to the loci generated by M_2 and M_3.*

The system can be generated as the intersection of three special systems of ∞^5 lines formed by tangents to three varieties of three dimensions; *i. e.*, three hypersurfaces.

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[*] In the *Rendi Conti di Palermo*, 1890, Segre has demonstrated the general theorem that *through each line of a system of lines depending on* $n-1$ *parameters in a space of n dimensions will pass* $n-1$ *developables, and each line of the system touches* $n-1$ *focal varieties of* $n-1$ *dimensions.* Following the procedure above, this general theorem will be demonstrated later.

Either two or three of the points in which Π cuts Φ may coincide. In the first case ϕ_3', ϕ_3'' coincide, say; then all the focal planes are tangent to ϕ_3', but only the planes μ_1 are tangent to ϕ_3'''. If all three points in which Π cuts Φ coincide, there is only one focal variety, and the focal planes are all tangent to this variety. This system is formed by the tangent lines to ∞^2 curves traced on a hypersurface whose osculating planes are tangent planes to the hypersurface.

Singular Cases. 1. The plane Π may coincide with one of the planes (7), in which case *on r there is one focus, but each of the ∞^2 planes passing through r is a focal plane;* or

2. The plane Π may cut Φ in a line of (8), in which case *there is one focal plane, but every point of r is a focal point;* or

3. Π may cut one of the planes of (7) in a line, in which case *one point of r is a focus, but ∞^1 planes of a pencil are focal planes;* or

4. Π may cut one of the planes of (7) in a line and contain one of the lines of (8). In this case *there is one focus, but ∞^1 focal planes* (corresponding to the lines of (8) which cut the line in which Π cuts the plane of (7)); *but corresponding to the line of* (8) *we have one focal plane and ∞^1 focal points.* Thus we see that the plane and point corresponding to the lines in which Π cuts Φ play a sort of double rôle.

5. The plane Π may cut Φ in a conic not lying in a plane of (7); for an S_3 which contains two lines of (8) would intersect Φ in a quadric surface, and hence, if Π lies in such an S_3, it would cut Φ in a conic. In this case *each point of r is a focus and ∞^1 planes passing through r are focal planes.* This differs from the preceding case, because there is a (1, 1) correspondence between the foci and focal planes; that is, to each focus is associated a definite focal plane. The focal planes form a plane pencil. An example of this is furnished by a complex of lines in a space of three dimensions.

The above conditions are not, in general, satisfied identically, but occur only for particular lines of the system. An example of lines of class 1 is the complex of lines through a point. In this complex each line is of class 1; that is, the relation is an identity. Another example is furnished by the lines tangent to the curve of intersection of ϕ_3', ϕ_3'', ϕ_3'''. The point of tangency is the point of contact, and the focal planes four in number are the three planes tangent to the three varieties ϕ_3', ϕ_3'', ϕ_3''' and the osculating plane of the curve. But we say that there can be only three focal planes unless Π cuts Φ in a particular manner, in which case there must be an infinite number of focal planes. There is one essential difference between these two examples. In the first, through each line

pass an infinite number of developables, while in the second, only four. In this second case, strictly speaking, the lines tangent to the curve should be counted as a part of the envelope of the focal planes, but following the custom of congruences of lines in S_3, we shall not include it here.

If the three varieties ϕ_3', ϕ_3'', ϕ_3''' have a common developable of tangent planes, then the generators of this developable are examples of lines of class 2. On each line there are four foci, the three points of contact with the three varieties and the point on the edge of regression of the developable. But if more than three points of r are foci, then all the points are foci. In this case also, strictly speaking, this developable should be counted as a part of the locus of the focal points, but here again we shall not consider it as such.

An example of a system of class 3 is furnished by the system of lines tangent to a surface. In this case the osculating planes to the curves which are traced on the surface, and which have a given tangent line r, form the focal planes of the system. These osculating planes form a plane pencil.*

Specializations. One of the focal hypersurfaces may become a surface. The system then consists of all the lines which cut a surface and are tangent to two hypersurfaces. The developables corresponding to the *focal surface* consist of the cone of lines whose vertex is on the surface and tangent to the two focal hypersurfaces. In the same manner two or three of the focal hypersurfaces may become surfaces. If one of the focal varieties reduces to a curve, it is at once seen that two of the focal varieties must coincide in this curve. The system then consists of all the lines which cut a curve and either are tangent to a hypersurface or cut a surface.

Systems of ∞^4 Lines, C_4.

If the coefficients a_1, a_2, ... are functions of four parameters u_1, u_2, u_3, u_4, the coordinates of the correlation are

$$(U) \begin{cases} a_1' = \dfrac{\partial a_1}{\partial u_1} \dfrac{du_1}{dt} + \dfrac{\partial a_1}{\partial u_2} \dfrac{du_2}{dt} + \dfrac{\partial a_1}{\partial u_3} \dfrac{du_3}{dt} + \dfrac{\partial a_1}{\partial u_4} \dfrac{du_4}{dt}, \\[2mm] a_2' = \dfrac{\partial a_2}{\partial u_1} \dfrac{du_1}{dt} + \dfrac{\partial a_2}{\partial u_2} \dfrac{du_2}{dt} + \dfrac{\partial a_2}{\partial u_3} \dfrac{du_3}{dt} + \dfrac{\partial a_2}{\partial u_4} \dfrac{du_4}{dt}, \\[2mm] \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots, \\[1mm] \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots, \\[1mm] a_3' = \dfrac{\partial a_3}{\partial u_1} \dfrac{du_1}{dt} + \dfrac{\partial a_3}{\partial u_2} \dfrac{du_2}{dt} + \dfrac{\partial a_3}{\partial u_3} \dfrac{du_3}{dt} + \dfrac{\partial a_3}{\partial u_4} \dfrac{du_4}{dt}. \end{cases}$$

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e Segre: "Su una classe di superficii, ecc.," p. 7.

We have seen that the points of the line r are represented by the planes of the system (7) and that the planes passing through r are represented by the lines of the system (8). Every Chasles correlation establishes a (1, 1) correspondence between the planes of (7) and ∞^1 lines of (8), but evidently not every correspondence of this sort represents such a correlation. In the first place it would be necessary for the ∞^1 planes through r which correspond to the ∞^1 lines of (8) to generate an S_3. Any S_3 in Σ cuts the variety Φ in a cubic curve, and thereby a (1, 1) correspondence is established between the points of r and ∞^1 planes passing through r; but not every such correspondence represents a Chasles correlation, for in Σ there are ∞^8 spaces S_3, but there are only ∞^6 Chasles correlations. In order to determine which of these cubics represent a Chasles correlation, we will first find the condition in order that two correlations be in involution.

By equation (2) a projectivity is established between the hyperplanes ρ_1 and the points z. Equation (5) is the condition in order that this projectivity should degenerate; hence, in order that two projectivities $(a_1', a_2', a_3', \alpha_1', \alpha_2', \alpha_3')$ and $(\bar{a}_1', \bar{a}_2', \bar{a}_3', \bar{\alpha}_1', \alpha_2', \alpha_3')$ be in involution the points in the space Σ represented by them must be conjugate with respect to the quadric (5). The same reasoning will show that the points should also be conjugate with respect to the quadric (6). Therefore the two projectivities are in involution when

$$a_1' \bar{a}_2' + \bar{a}_1' \alpha_2' - a_2' \alpha_1' - \bar{a}_2' \alpha_1' = 0, \tag{9}$$

$$a_1' \bar{a}_3' + \bar{a}_1' \alpha_3' - a_3' \bar{a}_1' - \bar{a}_3' \alpha_1' = 0. \tag{10}$$

Then all the projectivities which are in involution with a given projectivity lie in the S_3 formed by the intersection of (9) and (10). This S_3 cuts Φ in a cubic, and the points of this cubic represent the degenerate projectivities which are in involution with the two given ones; but in order that a non-degenerate projectivity should be in involution with a degenerate one, it is necessary that the singular point and the singular plane of the degenerate projectivity should be corresponding elements of the non-degenerate one. Therefore we see that the correlation corresponding to any point of Σ is the same as that which is set up by the points of the cubic of intersection of Φ with the polars of the point with respect to the two quadrics (5) and (6). Hence we have the theorem: *The cubics which determine a Chasles correlation are the intersections of Φ with the polars of the points of Σ with respect to the quadrics (5) and (6).*

The space U which represents the correlations belonging to C_4 cuts Φ in a cubic, and therefore a (1, 1) correspondence is established between the points of r and ∞^1 planes passing through r; but as we saw, these planes do not, in general, generate an S_3. To find the locus of these planes, we need only find the relation established between ρ_1 and ρ_2, and then from equations (2) and (3) find the equation of the locus. Let the S_3 be given by the equations

$$\Sigma A_i a_i' = 0, \qquad \Sigma B_i a_i' = 0.$$

Then the condition that this S_3 cut a line of (8) is

$$(B) \qquad \begin{vmatrix} 1 & -\rho_1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & -\rho_1 & 0 \\ 1 & 0 & -\rho_2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & -\rho_2 \\ A_1 & A_2 & A_3 & A_4 & A_5 & A_6 \\ B_1 & B_2 & B_3 & B_4 & B_5 & B_6 \end{vmatrix} = 0.$$

This is the relation sought between ρ_1 and ρ_2. It is easily seen from this determinant that there is no constant term or terms of the first degree in ρ_1 and ρ_2. The determinant when expanded is of the form

$$(C) \qquad \rho_1^2 \rho_2^2 + a\,\rho_1^2 \rho_2 + b\,\rho_1 \rho_2^2 + c\,\rho_1 \rho_2 + d\,\rho_1^2 + e\,\rho_2^2 = 0,$$

and substituting the values of ρ_1, ρ_2 from equations (2) and (3), we have the cone

$$x^2 + a\,xz + b\,xy + c\,yz + d\,z^2 + e\,y^2 = 0,$$

where

$$x = x_1 - a_1 - \alpha_1 x_4,$$
$$y = x_2 - a_2 - \alpha_2 x_4,$$
$$z = x_3 - a_3 - \alpha_3 x_4.$$

The condition that this cone should factor is the vanishing of the discriminant. Hence, *in C_4 there are ∞^3 lines whose cone of singular planes factors into two ordinary spaces S_3.* These two spaces have a plane Π in common.

If the space U is the conjugate of a point with respect to Φ, that is, if it is of the kind described above, the (1, 1) correspondence becomes a Chasles correlation; this, however, imposes three conditions. Hence, *in C_4 there are ∞^1 lines (i. e., a ruled surface) such that the (1, 1) correspondence established by U between the points of r and ∞^1 planes passing through r, becomes a correlation of Chasles.*

Singular Cases. 1. The space U may be tangent to Φ, in which case the intersection will evidently consist of the plane and line of Φ which pass through the point of tangency. The space U is evidently tangent to (5) and (6) in the same point, and consequently the curve of intersection represents a Chasles correlation corresponding to the point of contact. Hence, *there is one point and one plane which belongs to each correlation on r* (this point and plane correspond to the plane and line in which U cuts Φ). For consider any point Q' of U; its polar with respect to (5) and (6) will evidently pass through the point of contact Q of U, and consequently Q represents a point of r and its corresponding focal plane. Hence, *all the ruled surfaces which pass regularly through such a line r are tangent in this point and have this plane for tangent plane.*

If U is tangent to Φ, five conditions are imposed; hence, in C_4 there are, in general, no lines of the kind mentioned above.

2. (a) U may cut Φ in a line of (8). The cubic of intersection then degenerates into this line and a conic, in which case the $(1, 1)$ correspondence between the points of r and ∞^1 planes passing through r degenerates into two parts: (1) The line cuts each plane of (7); hence, there is one focal plane, but each point of r is a focus corresponding. (2) The correspondence set up by the conic is still $(1, 1)$. In this case the focal planes passing through r generate an S_3; for in the determinant (B) all the minors of order 5 must vanish, in particular the minor formed by canceling the second row and fifth column, which gives the following relation:

$$a\,\rho_1\rho_2 + b\,\rho_1 + c\,\rho_2 = 0;$$

and hence the locus of the planes is an S_3.

(b) The space U may cut Φ in two lines of (8); then it will cut Φ in a regulus. In this case the correlations which belong to any triply infinite system contained in C_4 are represented by a plane which cuts Φ in a conic, and the pole of the plane with respect to the quadric will be a correlation in involution with all the correlations belonging to the triply infinite system. Within a system like this there is a surface of singularities, and the whole behavior is like that of the lines in an S_3.

Fivefold Infinite Families of Lines, C_5.

The coefficients are now functions of five parameters, and therefore the correlations contained in C_5 are represented in Σ by an S_4 cutting the hyper-surface Φ in a cubic surface standing in the S_4. Every cubic surface in a space

of four dimensions must be a ruled surface having a rectilinear directrix. In this case the directrix must necessarily be a line of (8). For the lines of Φ which cut a line lying in one of the planes (7) will generate a quadric, because the lines of (8) set up a (1, 1) correspondence between the planes of (7), and therefore to a line in one plane will correspond a line in any other plane, and the lines joining corresponding points of these two lines will generate a quadric. The S_4 cuts each plane of (7) in a line; and therefore each point of r is a focus, and corresponding to any point there are ∞^1 planes passing through r which are focal planes. These planes correspond to the lines of (8) which cut the line in which S_4 cuts the plane of (7) which corresponds to the given point. The planes which have any given point for focus generate an S_3, and all these spaces S_3 have a plane in common (corresponding to the directrix of the cubic surface). Between the points of r and these spaces S_3 there is then a (1, 1) correspondence. This correspondence may degenerate; then the spaces S_3 coincide. This would necessitate that the lines which cut any line which S_4 cuts from a plane of (7) would cut the lines which S_4 cuts from the other planes. If this happens, then it is evident that the cubic surface will degenerate. S_4 is then tangent to Φ. The surface then degenerates into a quadric and a plane of (7); then there is one point of r (corresponding to the plane just mentioned) to which every plane through r corresponds. Also, when we consider the quadric, we see that each point of r has the same ∞^1 planes for corresponding focal planes. In any system C_5 there is a C_4 of lines of this kind, since the condition that S_4 be tangent to Φ imposes one condition. We may designate the lines of this C_4 by the term *singular lines*.

The lines of C_5 which pass through a given point generate a cone of three dimensions. The planes tangent to this cone in any given line r form an ordinary pencil with r as axis (i. e., generate a space S_3), and the cones whose vertices all lie on a line r of the system have one common tangent plane. If the line r is such that the space S_4 is tangent to Φ (i. e., if r is a singular line), then the cones whose vertices lie on this line have an infinite number of common tangent planes. There are ∞^4 such lines. In the general case a plane tangent to the cone is a plane which contains two infinitely near lines of the system C_5; hence, from the above we see that *through each line r of C_5 there passes one plane which contains ∞^1 lines of the system infinitely near to r.*

In any hyperplane S_3 of S_4 there are ∞^3 lines which belong to any C_5; that is, an S_3 cuts an ordinary complex of lines from C_5.

Generalization.

If we consider a ruled surface in a space of n dimensions, and use equations analogous to (1), the equations which define the correlation between the points of a generator r and the tangent planes of the surface passing through r are of the form:

(1)
$$\frac{x_1 - a_1 - \alpha_1 x_n}{x_2 - a_2 - \alpha_2 x_n} = \frac{a_1' + a_1' z}{a_2' + a_2' z} = \rho_1,$$

(2)
$$\frac{x_1 - a_1 - \alpha_1 x_n}{x_3 - a_3 - \alpha_3 x_n} = \frac{a_1' + x_1' z}{a_3' + a_3' z} = \rho_2,$$

$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots,$$
$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots,$$

$(n-2)$
$$\frac{x_1 - a_1 - \alpha_1 x_n}{x_{n-1} - a_{n-1} - \alpha_{n-1} x_n} = \frac{a_1' + a_1' z}{a_{n-1}' + a_{n-1}' z} = \rho_{n-2}.$$

The coordinates of the correlation are

$$a_1', \ a_2', \ a_3', \ \ldots, \ a_{n-2}', \ a_{n-1}', \ a_1', \ a_2', \ a_3', \ \ldots, \ a_{n-2}', \ a_{n-1}',$$

where $a_1' = \dfrac{da_1}{dt}, \ \ldots, a_1, a_2, \ \ldots, \ \alpha_1, \alpha_2, \ \ldots$ all being assumed to be functions

of the parameter t. Then

(A)
$$\left\{ \begin{array}{l} a_1' a_2' - a_2' a_1' = 0, \\ a_1' a_3' - a_3' a_1' = 0, \\ \cdots\cdots\cdots\cdots, \\ \cdots\cdots\cdots\cdots, \\ a_1' a_{n-1}' - a_{n-1}' a_1' = 0 \end{array} \right.$$

are the conditions that the correlation should degenerate. Then, if we consider $a_1', a_2', \ldots, a_{n-1}', a_1', \ldots, \alpha_{n-1}'$ as the homogeneous coordinates of a point in space of $2n-3$ dimensions, the degenerate correlations are represented by the variety (A) of $n-1$ dimensions and of order $n-1$, since the space $a_1 = \alpha_1 = 0$ is common to all the cones (A). The points of a generator r of the ruled surface are represented by the linear spaces of $n-2$ dimensions:

$$a_1' + \alpha_1' z = 0,$$
$$a_2' + \alpha_2' z = 0,$$
$$a_3' + \alpha_3' z = 0,$$
$$\cdots\cdots\cdots,$$
$$\cdots\cdots\cdots,$$
$$a_{n-1}' + \alpha_{n-1}' z = 0,$$

where z is the parameter of the point on the generator. The planes passing through r are represented by the lines:

$$a_1' - \rho_1\, a_2' = 0,$$
$$a_1' - \rho_1\, a_2' = 0,$$
$$a_1' - \rho_2\, a_3' = 0,$$
$$a_1' - \rho_2\, a_3' = 0,$$
$$\cdots\cdots\cdots\cdots,$$
$$\cdots\cdots\cdots\cdots,$$
$$a_1' - \rho_{n-2}\, a_{n-1}' = 0,$$
$$a_1' - \rho_{n-2}\, a_{n-1}' = 0.$$

Now, for the properties of systems of lines in a space of n dimensions we can study the intersection of the variety (A) with the space which represents the correlations contained within the given system, just as in the preceding pages.

If we have an $(n-1)$-fold infinite system (that is, if $a_1, a_2, \ldots, a_{n-1}$, $\alpha_1, \alpha_2, \ldots, \alpha_{n-1}$ are functions of $n-1$ parameters), then

$$a_1' = \frac{\partial a_1}{\partial u_1}\frac{du_1}{dt} + \frac{\partial a_1}{\partial u_2}\frac{du_2}{dt} + \cdots + \frac{\partial a_1}{\partial u_{n-1}}\frac{du_{n-1}}{dt},$$
$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots,$$
$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots,$$
$$\alpha_{n-1}' = \frac{\partial \alpha_{n-1}}{\partial u_1}\frac{du_1}{dt} + \frac{\partial \alpha_{n-1}}{\partial u_2}\frac{du_2}{dt} + \cdots + \frac{\partial \alpha_{n-1}}{\partial u_{n-1}}\frac{du_{n-1}}{dt}.$$

The correlations contained in this system are therefore represented by a space of $n-1$ dimensions. Now, a space of $n-1$ dimensions will intersect the variety (A) in $n-1$ points. Hence, we have the theorem due to Segre (*Rendi Conti di Palermo,* 1890): *In S_n an $(n-1)$-fold infinite system of lines has $n-1$ focal hypersurfaces.* Using the same reasoning as in S_4, we see that each focal plane is tangent to all the focal hypersurfaces except that one which contains the focus corresponding to this plane.

Application to the Geometry of the Circle in Space.

If the equation of the sphere be written in the form

$$t(x^2 + y^2 + z^2) - 2ax - 2by - 2cz + 2d = 0, \tag{11}$$

the ratio of the five quantities t, a, b, c, d will determine any sphere in space. The spheres of ordinary space can then be represented by points in a space of four dimensions. The radius of (11) is

$$\rho^2 = \frac{a^2 + b^2 + c^2 - 2dt}{t^2},$$

from which we see that the point spheres (spheres of zero radius) are represented by a quadric in S_4,

$$\phi = a^2 + b^2 + c^2 - 2\,d\,t = 0. \tag{12}$$

The planes of ordinary spaces are represented by the hyperplane

$$t = 0.$$

This hyperplane is tangent to ϕ in the point $(0, 0, 0, 0, 1) \equiv \omega$ which corresponds to the plane at infinity in ordinary space. The lines in the plane at infinity in ordinary space correspond in S_4 to the lines in the hyperplane $t = 0$ which pass through the point ω. The hyperplane $t = 0$ intersects the quadric ϕ in an ordinary quadric cone K, vertex ω, the generators of which represent the lines in the plane at infinity tangent to the circle at infinity. The other minimum lines in S_3 are represented in S_4 by tangent lines to this cone K.

The angle between two spheres (t, a, b, c, d) and (t', a', b', c', d') is given by the relation

$$\cos \theta = \frac{a\,a' + b\,b' + c\,c' - d\,t' - d'\,t}{\sqrt{a^2 + b^2 + c^2 - 2\,d\,t}\ \sqrt{a'^2 + b'^2 + c'^2 - 2\,d'\,t'}}. \tag{14}$$

In S_4 this represents the anharmonic ratio between the two points (t, a, b, c, d), (t', a', b', c', d') and the two points in which the line joining them cuts the quadric ϕ.

If the spheres are orthogonal,

$$a\,a' + b\,b' + c\,c' - d\,t' - d'\,t = 0; \tag{15}$$

and hence we see that the equation

$$a\,u_1 + b\,u_2 + c\,u_3 + d\,u_4 + t\,u_5 = 0 \tag{16}$$

represents all the spheres which cut the fixed sphere $(-t, a, b, c, -d)$ orthogonally. Then in S_4 the hyperplane represents all the spheres orthogonal to a given sphere.

It is seen immediately from (11) that the spheres whose coordinates are

$$(\lambda t + \mu t',\ \lambda a + \mu a',\ \lambda b + \mu b',\ \lambda c + \mu c',\ \lambda d + \mu d')$$

are the spheres of the pencil determined by the two spheres (t, a, b, c, d), (t', a', b', c', d'); or, if we consider the envelope of these spheres, we may say that this represents a circle. But in S_4 this represents a line. Hence, the circles in ordinary space are represented by lines in S_4. If we consider the line as defined by the two points in which it cuts ϕ, we have the representation of circles due to Laguerre.*

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* *Œuvres,* Vol. II, p. 54.

Two spheres are tangent if the line which represents their circle of intersection is tangent to ϕ. The circle determined by two tangent spheres is a point circle. Therefore the point circles of S_3 correspond to the lines of S_4 which are tangent to ϕ. In particular, the lines which lie in ϕ represent those circles the spheres of whose pencil are all point spheres. Therefore, *the lines of ϕ correspond to the point circles composed to two coincident minimum lines.*

From (16) it was seen that the spheres of a hyperplane represent spheres orthogonal to a fixed sphere (represented by the pole of the hyperplane with respect to ϕ). But if two spheres are orthogonal to a fixed sphere, their circle of intersection is also orthogonal to this sphere. Then the circles of S_3 which are orthogonal to a fixed sphere are represented in S_4 by the lines in an S_3. Thus we have at once the relation between the ordinary line geometry and the geometry of circles orthogonal to a given sphere.*

Circles which lie on the same sphere (*i. e.*, which intersect in two points) are represented by lines which intersect in S_4. Annular surfaces and circled surfaces in S_3 are represented in S_4 by developable and ruled surfaces respectively.

In S_4 the plane is the intersection of two hyperplanes; hence a plane represents the circles which pass through two fixed points; viz., the points in S_3 represented by the points of contact of the two hyperplanes passing through the given plane and tangent to ϕ.

If a (1, 1) correspondence is established between the spheres of two pencils, the circles of intersection of corresponding spheres will generate a cyclide. Then in S_4 a quadric surface corresponds to a cyclide in ordinary space. Circled surfaces having two circle directrices which are cut by each circle in two points, are represented in S_4 by ruled surfaces having two right-line directrices, and therefore which stand in an S_3. Hence, such surfaces are composed of circles orthogonal to a fixed sphere.†

In this discussion we shall limit ourselves to configurations which properly stand in an S_4 and not in an S_3, since the latter are disposed of in the paper referred to.

All circles which are tangent to a fixed line in a fixed point are represented in S_4 by lines of a plane which is tangent to ϕ. Two circles which intersect in a single point are represented by lines in S_4 which determine an S_3 tangent to ϕ.

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* Moore: *Annals of Mathematics*, Series 2, Vol. VIII, p. 57. Forbes: "The Geometry of Circles Orthogonal to a Given Sphere," New York, 1904.

† See Moore: *Annals of Mathematics*, Series 2, Vol. VIII.

Circled Surfaces.

For interpretation in circle geometry we shall consider the following ruled surfaces: general developables, developables whose generators are tangent to ϕ, ruled surfaces such that the S_3 determined by two consecutive generators is tangent to ϕ, general ruled surfaces (i. e., surfaces such that the S_3 determined by two consecutive generators is not tangent to ϕ).

In S_4 there is one and only one line which cuts three given arbitrary lines.[*] Hence, on a ruled surface in space of four dimensions there is one curve which corresponds to the asymptotic lines in S_3; we will denote this curve by the term *asymptotic curve.* Two consecutive generators of a ruled surface will determine an S_3; in this way there are ∞^1 spaces of three dimensions determined by a ruled surface. Two such spaces S_3 infinitely near will intersect in a plane which contains the generator determining the S_3's and the tangent to the asymptotic curve at the point in which it cuts this generator. Hence, *on a circled surface there is one curve such that the circle C which cuts three consecutive generators of the surface in a point of this curve will also cut the same three generators again. Two consecutive generators of a circled surface determine an orthogonal complex of spheres, and two such consecutive complexes intersect in a congruence which is determined by C and the generator which it cuts.*

We saw that there was a projectivity between the points of a generator r of a ruled surface and ∞^1 planes, forming a pencil, which pass through r. Considering the tangent plane as the plane pencil of lines tangent to the surface in a given point of r, we have: *The pencils of tangent circles which lie on the spheres of the pencil determined by a circle C of a circled surface, all have the circle C in common; that is, each sphere which passes through C is tangent to the surface in two points of C.* All the lines tangent to a ruled surface in S_4 along a generator r lie in an S_3. Hence, all the tangent circles touching the circled surface in points of a generator are orthogonal to a fixed sphere; and therefore the lines joining the two points of contact of the various spheres passing through C and tangent to the circled surface, must pass through a fixed point O, center of the fixed sphere. Now the following theorem can be stated: *Each sphere which passes through a circle C of a circled surface is tangent to the surface in two points; and the lines joining these two points of contact all pass through the same point O, and there is a $(1, 1)$ correspondence between these lines and the spheres of the pencil.[†]*

.* Segre: *Rend: Conti di Palermo*, 1890.

† See Cosserat: "Sur le cercle considéré comme élément générateur de l'espace." Demartus: "Sur les surfaces à génératrice circulaire," *Annales de l'École Normale Supérieure*, 3rd series, Vol. II, p. 123.

In S_4 there is a quadric tangent to a ruled surface along a generator. Therefore in circled space there is a cyclide which is tangent along a circle generator of a circled surface and such that each circle of the cyclide cuts the given generator in two points; *i. e.*, has two-point contact with the circled surface. There are ∞^3 such cyclides.[*]

Enneper's classification of circled surfaces can be obtained here by considering the various relations which the S_3 determined by two consecutive generators can have with respect to the quadric ϕ.

Congruences of Circles.

Using homogeneous coordinates and letting

$$x_i = \phi_i(u, v) + t\psi_i(u, v), \qquad i = 1, 2, \ldots, 5,$$

be the equations of a line of the congruence, then the equations of the line infinitely near to this one will be

$$x_i = \phi_i(u, v) + \frac{\partial \phi_i}{\partial u} du + \frac{\partial \phi_i}{\partial v} dv + t\left(\psi_i(u, v) + \frac{\partial \psi_i}{\partial u} du + \frac{\partial \psi_i}{\partial v} dv\right),$$

and the S_3 which these two lines determine is that determined by the points

$$\phi_i, \quad \psi_i, \quad \frac{\partial \phi_i}{\partial u} du + \frac{\partial \phi_i}{\partial v} dv, \quad \frac{\partial \psi_i}{\partial u} du + \frac{\partial \psi_i}{\partial v} dv;$$

that is,

$$\Sigma A_i x_i = 0,$$

where the A_i are determinants from the matrix

$$\left\| \phi_i, \quad \psi_i, \quad \frac{\partial \phi_i}{\partial u} du + \frac{\partial \phi_i}{\partial v} dv, \quad \frac{\partial \psi_i}{\partial u} du + \frac{\partial \psi_i}{\partial v} dv \right\|.$$

This hyperplane will be tangent to ϕ if

$$\Sigma A_i^2 = 0.$$

[Here it is supposed that the equation of ϕ has been put into the sum of squares.] The A_i are of degree two in du, dv; hence in each congruence of lines there are four ruled surfaces such that consecutive generators determine an S_3 tangent to ϕ. Therefore in circled space we have : *Through each circle of a congruence pass four circled surfaces such that each generator cuts the generator infinitely near to it in one point.*[†]

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[*] See Cosserat, *loc. cit.*

[†] See Darboux: "Théorie des Surfaces," Vol. II, p. 5. Cosserat, *loc. cit.*

We can take the point of tangency of the S to represent the S_3 itself, and in this way we have four surfaces of ϕ connected with each congruence. In ordinary space the points of these surfaces represent the point spheres through which pass the two circles infinitely near; that is, these surfaces of ϕ represent the focal surfaces of the congruence.

' A congruence of spheres is represented in S_4 by a surface; and we saw that there were congruences of lines in S_4 composed of tangents to a surface along a system of curves, but such that the focal planes have no envelope. Hence, *a congruence of circles in which a circle intersects one infinitely near in two points is composed of the circle generators of a system of annular surfaces of a congruence of spheres.*

Now if the system of curves traced on the surface F in S_4 is a system of characteristics,[*] the tangents to these curves are also tangent to a second surface F' along a system of characteristics. (The tangents to any characteristic C of F touch F' along a characteristic C', but the lines are not tangent to C'.) Now the pencils of lines tangent to F' in two consecutive points of C' intersect in a line r tangent to C on F and to C_1 on F'. Then in S_3 the circle corresponding to such a line r is a circle of the annular surface corresponding to C and of the annular surface corresponding to C_1. Hence, *this circle is tangent to the focal surfaces of the congruences of spheres corresponding to F and F', and the focal surfaces of the congruence of circles corresponding to the lines tangent to F along the curves C are the four focal surfaces of the two congruences of spheres corresponding to F and F'.* The focal surfaces of any congruence of spheres can be made the focal surfaces of two such complexes of circles.

The two pencils of lines tangent to F' in two points infinitely near of C' intersect two by two on the line r. Therefore in the corresponding congruence of spheres in S_3 the two focal points of any sphere and the two focal points of a sphere infinitely near, corresponding to two points infinitely near of a characteristic C', lie on ∞^1 spheres of a pencil and hence on a circle of an annular surface corresponding to a characteristic of F. The two surfaces F and F' may be interchanged. The circle L corresponding to such a line r is called a *principal circle* of the congruence of spheres.[†]

Then we see that, in general, *a system of principal circles of a congruence G of spheres is also a system of principal circles for another congruence G', and the*

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[*] Segre: "Su una classe di superficii, ecc.," *Atti di Torino*, 1907.

[†] Darboux: "Théorie des Surfaces," Vol. II.

envelope of such circles is composed of the four focal surfaces of the two congruences of spheres G and G'.

If the surface F in S_4 is of the parabolic type, the two systems of characteristics coincide and the surfaces F and F' coincide, and the two pencils of tangent lines in points infinitely near, of a characteristic, have a line in common; hence, in the congruence of spheres corresponding, the principal circles are tangent four times to the focal surface of the congruence of spheres.

A futher specialization occurs when the surface F in S_4 is a ruled surface. This corresponds to a congruence of spheres whose focal surface is a circled surface; that is, the congruence consists of ∞^1 pencils of spheres. The properties of such a congruence were discussed when circled surfaces were discussed. The principal circles of such a congruence are the circles of the circled surface each of which touches the focal surface in each point (*i. e.*, lies on the focal surface).

We saw that, in general, there were not more than two focal points on each line or more than two focal planes through each line, and that the two focal points must coincide when the focal planes coincided, but that it might happen that there was one focal point and an infinite number of focal planes, or vice versa. The conditions do not occur in general for each line of the congruence. In congruences of circles, likewise, it may happen that there is a single focal sphere (corresponding to focal point) and an infinite number of focal point pairs (corresponding to focal plane in S_4), or vice versa. An example of an infinite number of focal point pairs and a single focal sphere would be obtained if two congruences have an annular surface of principal spheres (spheres corresponding to a characteristic on a surface in S_4). Then the congruence of principal circles will have the circles of this annular surface as singular circles of the kind described above. As in the case of lines in S_4 this does not mean that an infinite number of annular surfaces pass through a circle of this congruence.

Triply Infinite Systems of Circles, C_3.

We saw that in a three-parameter system of lines in S_4 there were three developables passing through each line; hence, in a three-parameter system of circles three annular surfaces composed of circles of the system pass through each circle of the system.[*] The lines of the system in S_4 touched three hypersurfaces; hence, the circles of a three-parameter family are composed of the circles of ∞^2 annular surfaces contained in a complex of spheres. Such a system

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[*] See Cosserat, *loc. cit.*

of annular surfaces is also contained in twc other complexes. The triply infinite system of circles can then be generated by the intersection of three systems of ∞^5 circles composed of the circles on each sphere of a complex where it is cut by the spheres infinitely close.

We saw that a plane in S_4 corresponds to all the circles which pass through two points. We may therefore take this pair of points in S_3 as representing the plane in S_4. The planes which pass through a line l will correspond to the point pairs on the corresponding circle. The hyperplane tangent to a hypersurface ψ in a point P will correspond to a linear complex of spheres tangent to the complex of spheres which correspond to ψ. To a plane tangent to ψ in P will correspond a point pair on the sphere s which corresponds to P. Hence on s there are ∞^2 such point pairs, and the lines of these point pairs all pass through the same point (center of the fundamental sphere of the tangent linear complex). All planes tangent to ψ and passing through a line l will correspond to ∞^1 point pairs on the corresponding circle C. *The lines of these point pairs form a pencil.* In S_4 we saw that the focal planes which pass through a given line l of C_3 were tangent to the three focal surfaces. Let us call p_1, p_2, p_3 the three focal planes corresponding to the focal points P_1, P_2, P_3, and in circled space let p_1, p_2, p_3 be the point pairs corresponding to p_1, p_2, p_3, and s_1, s_2, s_3 be the spheres which correspond to P_1, P_2, P_3; and let ψ_1, ψ_2, ψ_3 be the complexes of spheres generated by s_1, s_2, s_3 respectively. The plane p_1 was seen to be tangent to the locus of P_2, P_3; hence, in circled space the ∞^1 point pairs on the circle C (corresponding to l) determined by the two complexes ψ_2 and ψ_3 have a point pair in common, but do not have a pair in common with the ∞^1 determined by ψ_1. If these three pencils of point pairs have a pair in common, the complexes would be such that ψ_1 and ψ_2, for example, would coincide; for in S_4 the focal plane p_1 would be tangent to all three hypersurfaces, and we saw that this could occur only if two of the focal hypersurfaces should coincide. If this condition should happen for each singular point pair (point pair corresponding to a focal plane), then all three of the focal complexes would coincide.

Singular Cases. Corresponding to the singular cases of lines in S_4 we have the following singular cases for circles: (1) Each point pair on any circle may be singular, but all corresponding to the same singular sphere; *e. g.*, all the circles which lie on a sphere. (2) There may be only one singular point pair, but each sphere passing through the circle being a singular sphere. (3) One sphere may be singular and ∞^1 point pairs of a pencil (*i. e.*, the lines of the point

pairs form a pencil) correspond to it. (4) The singular spheres and point pairs may divide into two parts. In the first there is one singular sphere, but ∞^1 point pairs of a pencil correspond to it, while in the second there is one singular point pair, but each sphere is singular. (5) Each sphere is singular and there is a definite singular point pair corresponding to each sphere. These point pairs form a pencil. This differs from the preceding case in the fact that there is a (1, 1) correspondence between the singular spheres and the singular point pairs. An example of this kind of system is furnished by a complex of circles orthogonal to a fixed sphere.

Fourfold Infinite Systems of Circles, C_4.

The correlations between spheres and point pairs of a circle which belongs to a four-parameter family of circles form a linear system of three dimensions. Each sphere passing through the circle is a singular sphere and to each sphere corresponds a definite singular point pair.

In S_4 we saw that the singular planes generated a quadric cone having the given line of the system for vertex. To the singular plane in S_4 corresponds the singular point pair in circled space, and we shall now seek the envelope of the lines joining the singular point pairs corresponding to the spheres which pass through a given circle of the system.

The line in circled space which joins the points of a point pair (*i. e.*, the line joining the points through which pass all the circles correspond to the lines in S which lie in the same plane π) are represented in S_4 by the line in which π cuts the hyperplane t whose lines represent lines in circled space instead of circles. For through the two points must pass one line and this line must be represented by the intersection of t and π. The quadric cone of three dimensions mentioned above cuts t in an ordinary quadric cone of two dimensions. For the vertex of the three-dimensional cone cuts t in one point and the planes of this cone cut t in lines which must pass through this point and therefore generate a quadric cone in t. A quadric cone in t corresponds to a conic in ordinary space; hence: *Each sphere passing through a circle of a four-parameter family is a singular sphere, and to each sphere corresponds a definite singular point pair. The lines joining the points of these singular point pairs envelope a conic.*

It imposes one condition on the parameters if the above conic should degenerate. Hence, in a four-parameter family of circles there is a three-parameter family such that the correspondence between singular spheres and singular point pairs degenerates. Such circles may be called *singular* circles.

We saw that in a four-parameter family of lines in S_4 there where ∞^1 lines such that the correspondence between singular points and singular planes was a Chasles correlation; then in circled space there are ∞^1 circles in a four-parameter family such that the lines of the singular point pairs form a plane pencil.

Singular Cases.   1.   There may be circles such that all the circled surfaces - which belong to the system and pass through one of these circles have a tangent sphere and corresponding point pair in common.   Such circles do not in general exist.

2.   There may be a circle C for which the correspondence between the focal spheres and focal point pairs will degenerate into two parts: (1) One focal point pair which corresponds to each sphere passing through C; (2) a $(1, 1)$ correspondence between the focal spheres and the focal point pairs.   The lines of the point pairs in the second case generate a plane pencil.

3.   There may be circles which behave like circles in a circled space formed by all the circles which cut a fixed sphere orthogonally.   If this condition is an identity, we have the geometry of circles orthogonal to a fixed sphere.

Fivefold Infinite Systems of Circles, C_5.

The correlations between spheres and point pairs which belong to a fivefold infinite system of circles form a linear system of four dimensions.

Each sphere which passes through a given circle C of the system is a focal sphere, and to each focal sphere correspond ∞^1 focal point pairs.   The lines of these point pairs generate a plane pencil and the ∞^1 pencils corresponding to the different spheres which pass through C have a line in common.   There is a $(1, 1)$ correspondence between the spheres passing through C and these pencils of lines.

In a five-parameter family of circles there is a four-parameter family such that the above correspondence between spheres and pencils of lines degenerates. In this case all the spheres passing through C correpond to the same pencil of lines.   Also to each of a simply infinite number of point pairs on C corresponds each sphere passing through C.   The lines of these point pairs form a plane pencil.

On the Geometry of Line Elements in the Plane with Reference to Osculating Circles.

By George F. Gundelfinger.

Introduction.

The position of a line element in the plane is fixed by three coördinates (x, y, p). If we regard these coördinates as functions of a single parameter t,

$$x = x(t), \quad y = y(t), \quad p = p(t),$$

we have an analytic definition of a *line-element locus of* ∞^1 *elements*. We associate with this line-element locus a *point locus* defined by

$$x = x(t), \quad y = y(t)$$

and a *line locus*, which is the envelope, of the one-parameter family of lines defined by

$$y - y(t) = p(t) [x - x(t)].$$

If the point locus and the line locus of a line-element locus coincide, we call the latter a *union*, the analytic condition for such being

$$dy - p\, dx = 0. \tag{1}$$

There are two particular unions of especial interest — the *point union* and the *line union* — for the singularities of a general union arise from its extraordinary relation to these two.[*]

Now, if we regard the coördinates of a line element as functions of two independent parameters

$$x = x(t, s), \quad y = y(t, s), \quad p = p(t, s), \tag{2}$$

we have a *line-element locus of* ∞^2 *elements*. Such a locus may always be resolved

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[*] Two unions are tangent when they have a line element in common, and osculate when they have two consecutive line elements in common. If a point union or a line union has more than one line element in common with a general union, that point or line is *singular* with respect to the general union; thus, for example, a double point or a double tangent; or if the two line elements are consecutive, a cusp or an inflectional tangent. That is, a cusp is a point which *osculates* a curve.

in infinitely many ways into ∞^1 loci of ∞^1 elements, but in one way only into ∞^1 unions. If we eliminate the two parameters from equations (2), we obtain in general[*] an *ordinary differential equation of the first order*,

$$f(x, y, p) = 0,$$

whose integral curves are the ∞^1 unions referred to. We associate then in general an ordinary differential equation of the first order with a line-element locus of ∞^2 elements.

The object of this paper is to obtain theorems on the two types of line-element loci just considered. The problem in the plane is changed to one in space by a transformation due to Lie,[†] viz.,

$$T \begin{cases} x = i\left(Z + \dfrac{X}{Y}\right), \\[2mm] y = Z - \dfrac{X}{Y}, \\[2mm] p = \dfrac{i(1 - Y^2)}{1 + Y^2}, \end{cases}$$

which sets up a correspondence between the points (X, Y, Z) of space and the line elements (x, y, p) in the plane. Hence, curves and surfaces give rise to line-element loci of ∞^1 and of ∞^2 elements respectively.

Furthermore, under the transformation T the equation (1) becomes

$$X\,dY - Y\,dX + dZ = 0, \tag{3}$$

and hence only those curves in space which are integral curves of this Pfaffian equation give rise to unions in the plane. But these curves are those of a linear line complex whose axis is the axis of Z; that is, the tangents of all such curves constitute an assemblage of ∞^3 lines whose Plückerian line coördinates satisfy a linear relation. These coördinates are defined by the two-rowed determinants of the matrix

$$\left\| \begin{array}{cccc} X & Y & Z & 1 \\ dX & dY & dZ & 0 \end{array} \right\|,$$

viz.,

$$\begin{array}{ll} p_1 = Z\,dY - Y\,dZ, & p_4 = dX, \\ p_2 = X\,dZ - Z\,dX, & p_5 = dY, \\ p_3 = Y\,dX - X\,dY, & p_6 = dZ, \end{array} \tag{4}$$

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[*] Unless the ∞^1 unions are all point unions.

[†] "Geometrie der Berührungstransformationen," p. 247. The transformation given above differs slightly from Lie's transformation by the projective transformation

$$\bar{X} = 4X; \quad \bar{Y} = \frac{Y}{2}; \quad \bar{Z} = 2Z,$$

the change being made to facilitate the analysis.

which satisfy the identical relation

$$p_1 p_4 + p_2 p_5 + p_3 p_6 \equiv 0. \tag{5}$$

The linear equation which the line coördinates satisfy is with reference to (3) and (4) easily seen to be

$$p_3 = p_6.$$

We shall hereafter refer to this as the fundamental complex Γ. The *aequatio directrix* of the equivalent null-system is

$$Y_0 X - X_0 Y - Z + Z_0 = 0. \tag{7}$$

The geometry in space which we shall interpret in the plane by means of the transformation T is the projective geometry within the fundamental complex Γ; that is, the geometry whose group is the G_{10} of all linear transformations under which the ∞^3 lines of Γ form an invariant system, or in other words, all linear transformations which leave the equation (3) invariant.

It will be shown later[*] that under T the lines of Γ transform into *circular unions* in the plane, and we have already stated that equation (3) transforms into (1). Hence, the geometry in the plane is that geometry whose group is the Γ_{10} of all contact transformations which leave the system of ∞^3 circles in the plane invariant. Knowing then the distinct types of curves and surfaces under G_{10} in space, we are in a position to classify line-element loci of ∞^1 and of ∞^2 elements with respect to Γ_{10} in the plane.

§ 1. *Line-Element Loci of ∞^1 Elements.*

Consider a general skew curve in space defined by

$$X = f(t), \qquad Y = g(t), \qquad Z = h(t).$$

Then, under the transformation T,

$$x = i\left(Z + \frac{X}{Y}\right),$$

$$y = Z - \frac{X}{Y},$$

$$p = \frac{i(1 - Y^2)}{1 + Y^2},$$

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[*] See also "Geometrie der Berührungstransformationen."

the corresponding line-element locus in the plane is

$$x = i\left(h + \frac{f}{g}\right),$$

$$y = h - \frac{f}{g},$$

$$p = \frac{i(1 - g^2)}{1 + g^2}.$$

The point locus is defined parametrically by the first two of these equations, and the line locus is defined by the envelope of the family of lines

$$(1 + g^2)\,y - i\,(1 - g^2)\,x = 2\,(h - fg),$$

which envelope has the following parametric equations:

$$x = i\left(h + \frac{f}{g}\right) - \frac{i\,(1 + g^2)}{2\,gg'}\,(fg' - gf' + h'),$$

$$y = h - \frac{f}{g} + \frac{1 - g^2}{2\,gg'}\,(fg' - gf' + h').$$

The equations defining the point locus and the line locus are identical when and only when

$$fg' - gf' + h \equiv 0,$$

which is the condition that the original skew curve be a curve of the complex Γ, or an integral curve of equation (3). This agrees with our statement that only those curves of Γ give rise to unions in the plane. We shall refer to such curves hereafter simply as *complex* curves.

THEOREM 1: *The complex curves of space and such curves only give rise to unions in the plane.*

Consider now a general line in space. Its six coördinates enter into the equations of its projecting planes as follows:

$$\left.\begin{array}{l} p_6\,X = p_4\,Z + p_2, \\ p_6\,Y = p_5\,Z - p_1, \\ p_5\,X = p_4\,Y - p_3. \end{array}\right\} \quad (8)$$

The equations of the line may then be written in the following parametric form:

$$\left.\begin{array}{l} X = \dfrac{p_4\,t - p_3}{p_5}, \\[2mm] Y = t, \\[2mm] Z = \dfrac{p_6\,t + p_1}{p_5}. \end{array}\right\} \quad (9)$$

The corresponding line-element locus is

$$x = \frac{i}{p_5 t} [p_6 t^2 + (p_1 + p_4) t - p_3],$$

$$y = \frac{1}{p_5 t} [p_6 t^2 + (p_1 - p_4) t + p_3],$$

$$p = i \left[\frac{1 - t^2}{1 + t^2}\right].$$

The point locus, found by eliminating the parameter from the first two equations, has the equation

$$p_5^2 (x^2 + y^2) - 2 i p_5 (p_1 + p_4) x - 2 p_5 (p_1 - p_4) y + 4 p_2 p_5 = 0.^* \tag{10}$$

The line locus is the envelope of the family of lines

$$[p_5 (y + i x) + 2 p_4] t^2 - 2 (p_3 + p_6) t + [p_5 (y - i x) - 2 p_1] = 0.$$

Eliminating t from this equation and its derivative, we obtain

$$p_5^2 (x^2 + y^2) - 2 i p_5 (p_1 + p_4) x - 2 p_5 (p_1 - p_4) y - [(p_3 + p_6)^2 + 4 p_1 p_4] = 0. \tag{11}$$

Hence,

THEOREM 2: *The point locus and the line locus of a general line in space are both circles having the same center.*

Since the coördinates of the reciprocal polar line under Γ of a given line are found by interchanging the p_3 and p_6 coördinates of the latter, we can state by reference to (10) and (11)

THEOREM 3: *Reciprocal polar lines of Γ have the same point locus and the same line locus.*

Setting $p_3 = p_6$, the general line becomes a complex line and equations (10) and (11) both reduce † to

$$p_5 (x^2 + y^2) - 2 i (p_1 + p_4) x - 2 (p_1 - p_4) y + 4 p_2 = 0. \tag{12}$$

THEOREM 4: *There is a one-to-one correspondence between the complex lines in space and the circular unions in the plane.*

Consider those complex lines which are perpendicular to the axis of Z. They are defined by

$$Z = k,$$
$$X = K Y. \tag{13}$$

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* The constant term has been simplified by means of (5).

† The constant term in (11) becomes $- (4 p_6^2 + 4 p_1 p_4) = 4 p_2 p_5$, by (5).

Hence, the corresponding unions are characterized by

$$x = i\,(k + K) = \text{const.,}$$
$$y = k - K = \text{const.,}$$

evidently point unions. We shall refer to such lines as P lines.

THEOREM 5: *There is a one-to-one correspondence between the P lines in space and the point unions in the plane.*

Consider those complex lines which are perpendicular to the axis of Y. They are defined by

$$Y = h,$$
$$Z - h\,X = H. \tag{14}$$

The corresponding unions are characterized by

$$p = \frac{i\,(1 - h^2)}{1 + h^2} = \text{const.,}$$

evidently line unions. Hence, we shall refer to these lines as L lines.

THEOREM 6: *There is a one-to-one correspondence between the L lines in space and the line unions in the plane.*

We tabulate this correspondence as follows:

Space.	*Plane.*
Point.	Line element.
Complex curve.	Union.
Complex line.	Circular union.
P line.	Point union.
L line.	Line union.

Consider now two general curves in space. If these curves intersect, their corresponding line-element loci have a line element in common; hence

THEOREM 7: *If two general curves in space intersect, their corresponding point loci intersect and their corresponding line loci have a common tangent.*

Let the two general curves be tangent; then their corresponding line-element loci have two consecutive line elements in common; hence

THEOREM 8: *If two general curves in space are tangent, their corresponding line loci as well as their corresponding point loci are also tangent.*

Consider two complex curves. If they intersect, their corresponding unions have a line element in common.; hence

THEOREM 9: *If two complex curves intersect, their corresponding unions are tangent.*

Apply this theorem to the interpretation of all the complex lines lying in a given plane. Since one of these lines is a P line and one an L line, and since all pass through a common point, we state

THEOREM 10: *Corresponding to all the complex lines lying in a given plane, we have a pencil of tangent circular unions, including the point union and the line union determined by the common line element.*

If two complex curves are tangent, their unions have two consecutive line elements in common; hence

THEOREM 11: *If two complex curves are tangent, their corresponding unions osculate;* and, in general,

THEOREM 12: *If two complex curves have contact of the n-th order, their unions have contact of the $(n+1)$-st order.*

Applying theorems 11 and 12 to a complex curve and its tangents, we obtain the following interesting correspondence:

THEOREM 13: *A complex curve and its tangents have the following interpretation in the plane:*

Complex curve.	Union.
∞^1 tangents.	∞^1 osculating circles.
P tangent.	Cusp.*
L tangent.	Inflectional tangent.*
Stationary tangent.†	Hyperosculating circle.‡

Knowing the distinct types of curves under G_{10},§ we obtain the following classification of line-element loci of ∞^1 elements with respect to Γ_{10}.§

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* Note that the simple point and line singularities on a union arise not from singularities on the corresponding complex curve in space, but from the relation of the latter to the P and L lines of the complex. For higher singularities on the union, however, the complex curve itself must possess singularities. For example, a node on the complex curve gives rise to a tac-node on the union.

† The condition for a stationary point (cusp) on a complex curve is given by $g'f'' - g''f' = 0$. See Picard, *Annales de l'École Normale Supérieure*, 2. série, 6.

‡ Note that a cusp on a complex curve does not in general give rise to a singularity, but an extraordinary point (vertex) on the union, unless the cuspidal tangent is a P line or an L line, in which case the union possesses higher point or line singularities.

§ See Introduction.

Space.	*Plane.*
I.	**I.**
The general skew curve.	*Skew* [*] line-element locus.
II.	**II.**
The general plane curve. (See Th. 10.)	*Plane* [*] line-element locus; that is, a general line-element locus of ∞^1 elements selected from a pencil of tangent circular unions.
III.	**III.**
The general line. (See Th. 2.)	*Linear* [*] line-element locus; that is, one whose point locus and line locus are concentric circles.
IV.	**IV.**
The general complex curve. [†] (See Th. 1.)	The general union.
V.	**V.**
The complex line. [‡] (See Ths. 4, 5, 6.)	Circular, point and line unions. [§]

§ 2. *Line-Element Loci of ∞^2 Elements.*

Consider a general surface in space defined by

$$F(X, Y, Z) = 0.$$

There is one complex line tangent to this surface at each point; viz., the intersection of the tangent plane and the complex plane [||] at that point. There are then ∞^1 complex curves on a surface. The line of intersection becomes indeterminate in two cases:

(a) when the tangent plane is indeterminate; [¶]

(b) when the tangent plane and the complex plane coincide.

The first case is that of surface singularities, which we shall not consider in detail at this point, since it bears no especial relation to the geometry within the complex. [**]

---

[*] The various types are named after the corresponding curves in space.

[†] No complex curve except the straight line can be plane.

[‡] P and L lines are not in themselves distinct under G_{1C}.

[§] These three types are not individually distinct.

[||] That is, the plane determined by the null-system. See Introduction, (7).

[¶] The complex plane is never indeterminate.

[**] It is to be remarked that if a surface possesses an n-tuple curve, then n complex curves on the surface pass through each point on this curve.

The second case is the more important in the work that follows. A point on a surface where the tangent plane and complex plane coincide will be referred to as a *singular complex point*. It must be clearly understood that such points are not surface singularities, but are singular only with respect to the *position* of the surface in the complex. If such points are finite in number or infinite and at the same time discrete, we shall call them *complex nodes*. If the locus of singular complex points on a surface is a continuous curve, we shall call the latter a *singular complex curve* of the surface, for it is indeed complex.

It is evident that an infinite number of complex curves on a surface intersect at its complex nodes, and that through each point on a singular complex curve there passes one other complex curve of the surface.

We wish to consider now the loci of such points on a general surface, where the complex tangent lines are P lines and L lines respectively. We shall call these loci the P *and the* L *curves* respectively of the surface.

Let the surface be defined by

$$F(X, Y, Z) = 0.$$

Let $P_0(X_0, Y_0, Z_0)$ be a point on the surface.

$$F(X_0, Y_0, Z_0) = 0.$$

The tangent plane at this point is defined by

$$F_{X_0} \cdot X + F_{Y_0} \cdot Y + F_{Z_0} \cdot Z - (X_0 F_{X_0} + Y_0 F_{Y_0} + Z_0 F_{Z_0}) = 0,$$

and the complex plane by

$$Y_0 X - X_0 Y - Z + Z_0 = 0.$$

The pencil of planes through their line of intersection is defined by

$$(F_{X_0} + K Y_0) X + (F_{Y_0} - K X_0) Y + (F_{Z_0} - K) Z$$
$$+ (K Z_0 - X_0 F_{X_0} - Y_0 F_{Y_0} - Z_0 F_{Z_0}) = 0. \tag{15}$$

This line of intersection is a P line if, for some value of K, the equation (15) reduces to

$$MZ + N = 0.$$

Hence,

$$F_{X_0} + K Y_0 = 0,$$
$$F_{Y_0} - K X_0 = 0.$$

Eliminating K from these equations and omitting subscripts, we obtain

$$X F_X + Y F_Y = 0. \tag{16}$$

We shall call the locus of this equation the P *surface of* F. Its curve of intersection with F, which is not at the same time a singular curve* of F, is evidently the P curve on the surface.

---

* Note that the P surface passes through all the surface singularities of the original surface which are defined by $F_X = F_Y = F_Z = 0$. The complex tangents at such points are not in general P lines. Note also that equation (16) may reduce to an absurdity, in which case the P curve does not exist, or it may give an identity, in which case we have a ruled surface F all of whose generators are P lines. These remarks apply equally well to the L surface.

Similarly, we derive an equation

$$F_X + Y F_Z = 0, \qquad\qquad (17)$$

whose locus we shall call the *L surface of F*, and whose curve of intersection with *F*, which is not at the same time a singular curve, is the *L* curve of the surface.

Note that if both relations (16) and (17) are satisfied by the coördinates of a point on *F*, then the tangent plane and the complex plane at that point coincide (see (14)). Hence, the singular complex points on a surface are defined by the intersection of the three surfaces

$$\left.\begin{array}{l} F(X,\,Y,\,Z) = 0, \\ X F_X + Y F_Y = 0, \\ F_X + Y F_Z = 0. \end{array}\right\} \qquad (18)$$

THEOREM 14: *The P and L curves of a surface pass through its complex nodes.*

If the three surfaces (18) have in common a curve which is not a surface singularity of *F*, that curve is the singular complex curve of *F*.

THEOREM 15: *A surface in general has no singular complex curve.*

The interpretation is the following:

Space.	*Plane.*
General surface.	General diff. equation of 1st order.
$F(X,\,Y,\,Z) = 0.$	$f(x,\,y,\,p) = 0.$
∞^1 complex curves of F.	∞^1 integral curves of f.
∞^2 complex tangents to F.	∞^2 osculating circles of integral curves.
P surface: $X F_X + Y F_Y = 0.$	Diff. equation: $f_p = 0.$*
L surface: $F_X + X F_Z = 0.$	Diff. equation: $f_x + p f_y = 0.$*
P curve.	Point locus is locus of cusps on integral curves. †
L curve.	Line locus is envelope of inflectional tangents. ‡
Complex node.	Tac-point.
Singular complex curve.	Singular solution.§

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* For analysis see foot-note in § 3.

† Obtained by eliminating p from $f=0$ and $f_p=0$, the well-known p-discriminant locus. The line locus of the P curve is the envelope of cuspidal tangents.

‡ The point locus of the L curve is the locus of inflections.

§ The singular solution satisfies the three equations

$$f = 0, \qquad f_p = 0, \qquad f_x + p f_y = 0.$$

Note, however, that the P and L surfaces always intersect the original surface in its singular curves. A singular curve on a surface evidently gives rise to a tac-locus in the plane; the line elements along this tac-locus also satisfy the three equations above. Usually we think of the line elements of the singular solution alone as satisfying these relations.

Note that if the singular complex curve on a surface is a P line, the singular solution becomes a ray-point.

Since the geometry within the complex is projective, we shall consider first the complex characteristics of ruled surfaces and the corresponding line-element loci of ∞^2 elements in the plane. We shall return to the general differential equation later.

Since the tangent plane at any point on a ruled surface contains the generator through that point, it follows that the singular complex points on a ruled surface can lie only on those generators which are complex lines.[*] Let l be a complex generator; the complex planes as well as the tangent planes pass through l for all points on l. Hence, by the theory of involution, for two points at most on l, the two planes coincide. Having shown that the number of singular complex points on any complex generator is finite, we can state

THEOREM 16: *A general ruled surface can not have a singular complex curve.*

THEOREM 17: *There are two complex nodes on every complex generator of a general scroll.*

THEOREM 18: *There is one complex node on every complex generator of a general developable.*

Consider developable surfaces: First a torse; in passing over the cuspidal edge of a torse the complex curves on the surface must acquire a cusp.[†]

THEOREM 19: *The cuspidal edge on a general torse is the locus of stationary points on its complex curves.*

We shall now center our attention on ruled surfaces all of whose generators are complex lines; they are evidently complex scrolls, complex torses and planes (complex cones).

By Th. 17, we obtain

THEOREM 20: *There are two singular complex curves on every complex scroll.*[‡]

We shall refer to a complex scroll all of whose generators are P lines as a P scroll,[§] and to one all of whose generators are L lines as an L scroll.[||]

---

[*] There are always some complex generators on a general ruled surface. Let the surface be defined by the intersection of the two planes

$$p_6 z = p_4 z + p_2 , \qquad p_5 x = p_4 y - p_3 ,$$

where the p's are functions of a single parameter t. The equation in t, $p_3 = p_6$, determines the complex generators.

[†] Unless the tangent to the curve at that point is tangent also to the edge, which is extraordinary.

[‡] This theorem is due originally to Picard, *loc. cit.*

[§] P scrolls are defined by $X = Y f(Z)$. See (13), §1.

[||] L scrolls are defined by $Z - Y X = f(Y)$. See (14), §1.

One of the singular complex curves on an L scroll is the line at infinity in the XZ plane. We wish to consider complex scrolls both of whose singular complex curves are complex lines. Select two non-intersecting complex lines in space. The locus of complex lines meeting both of these lines is by the theory of one-to-one correspondence a quadric surface. Conversely, consider a complex quadric. The tangent planes at the two complex nodes on a complex generator cut the quadric each in a second generator, which must also be complex; hence

THEOREM 21 : *The two singular complex curves on every complex quadric are two generators of the other system.*

THEOREM 22 : *There is one singular complex curve on a complex torse — its cuspidal edge.* *

THEOREM 23 : *The complex point in a plane is a complex node.*

It is evident that a complex scroll in space must give rise to a differential equation in the plane whose integral curves are circles: an L scroll, in particular, to one whose integral curves are straight lines, that is, a Clairaut type; while a P scroll goes over into ∞^1 point unions. Hence, we may state

THEOREM 24 : *All space curves lying on the same P scroll have the same point locus.*

THEOREM 25 : *All space curves lying on the same L scroll have the same line locus.*

We associate with a general surface, a P scroll and an L scroll which touch the surface along its P curve and L curve respectively. Hence, they both touch the surface at the points where these curves intersect; viz., at the complex nodes. It follows then that these points are singular complex points on the scrolls themselves, and hence

THEOREM 26 : *The singular complex curves on the P and L scrolls of a surface pass through the complex nodes on the surface.*

Interpreted in the plane by means of Theorems 9, 24 and 25, this becomes

THEOREM 27 : *The locus of cusps on the integral curves of a differential equation touches the envelope of their inflectional tangents at the tac-points.*

We wish now to classify differential equations (line-element loci of ∞^2 elements) with respect to Γ_{10}. We shall first enumerate the distinct types of surfaces in space under G_{10}. In classifying surfaces, we must notice that while

---

* Note here the extraordinary case of the complex curves on a surface being tangent to its singular complex curve.

all complex lines in space form an invariant system, the P and the L lines are not in themselves distinct types, and properties of a surface depending on these lines are not invariant. On the other hand, the complex singularities — the complex node and the singular complex curve — are invariant properties. Ruled surfaces all of whose generators are complex lines, since they always possess singular complex curves, can be classified directly from this point of view.

Complex Ruled Surface.	*Diff. Eq. whose Int. Curves are Circles.*
VI.	VI.
General complex scroll with two general singular complex curves (IV).	General one-parameter family of circles with a double envelope.
VII.	VII.
Complex scroll one of whose singular complex curves is a complex line (V).	One-parameter family of circles touching a curve and a fixed circle.*
VIII.	VIII.
Complex quadric. (See Th. 21.)	All the circles touching two fixed circles.*
IX.	IX.
Complex torse. (See Th. 22.)	All the circles osculating a given curve.
X.	X.
Plane. (See Th. 10.)	Pencil of tangent circles.†

We know that the remaining distinct types of surfaces under G_{10} are the general surface, the general scroll, the general torse and the general cone. By Theorem 16, none of the last three can have a singular complex curve, while the general surface may or may not possess one. But the distinguishing characteristics of the corresponding differential equations in the plane are obtained from a consideration of the arrangement of the ∞^2 complex lines in space which are tangent to the surface. Since these correspond to the osculating circles of the integral curves in the plane, we see that the basis of classification is the distinct arrangements under Γ_{10} of the ∞^2 circles which osculate the integral curves, and hence the significance of the title of the paper is realized.

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* The two fixed circles may also be points or lines.

† Note that all the line elements through the tac-point satisfy the differential equation (see Th. 10).

Now if the surface is a developable, then the tangent complex lines lie in its ∞^1 tangent planes, which clears up the arrangement for the torse and the cone — the latter being distinguished from the former in that its tangent planes have a point in common. For the general surface the arrangement is general. The case of a general scroll needs more consideration. We fix our attention on one generator; the complex lines tangent to the surface along this generator meet also a second line (the reciprocal polar), and hence are the generators of a complex quadric.

XI.

General surface: Tangent lines have no particular arrangement.

XII.

General scroll: The ∞^2 complex tangent lines can be grouped as the generators of ∞^1 complex quadrics which touch the scroll each along a generator.[*]

(See Types III and VIII.)

XIII.

General torse: The ∞^2 complex tangent lines lie in ∞^1 planes.

(See Type X.)

XIV.

General cone: The ∞^2 complex tangent lines lie in ∞^1 planes having a point in common.

XI.

General differential equation: The osculating circles have no particular arrangement.

XII.

Scroll[†] differential equation: The ∞^2 osculating circles can be grouped into ∞^1 families. The circles of each family touch two other circles and osculate the integral curves along a circle. The tangents to the integral curves along the latter circle envelope a concentric circle.

XIII.

Torse differential equation: The ∞^2 osculating circles can be grouped into ∞^1 pencils of tangent circles.

XIV.

Cone differential equation: The arrangement is similar to that in XIII, with the specialization that there is one pencil having one circle in common with each of the other pencils.

The properties here given of these four types of differential equations sufficiently characterize them; but they have additional properties by far more interesting, reserved for the following section.

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[*] This last clause is very important, since the envelope of ∞^1 ruled surfaces is not necessarily a ruled surface.

[†] We shall name the differential equation after the corresponding surface.

§ 3. *Reciprocation.*

The linear line complex Γ is invariant also under the correlation which it defines when regarded as a null-system, for then each line of Γ is itself invariant. This fact gives rise to a theory of reciprocal line-element loci in the plane.

Let us recall the notion of reciprocation in space. Consider a general surface S of ∞^2 points and of ∞^2 tangent planes. Then under the null-system (7) we associate ∞^2 complex planes with the ∞^2 points on S; they envelope a surface $\bar{S}$ which is also the locus of the ∞^2 complex points of the tangent planes of S. We refer to S and $\bar{S}$ as reciprocal surfaces under Γ.

The points on S and $\bar{S}$ are in a one-to-one correspondence such that if P and $\bar{P}$ are corresponding points, the tangent plane to S at P is the complex plane of $\bar{P}$, and the complex plane of P is the tangent plane to $\bar{S}$ at $\bar{P}$. The line joining P and $\bar{P}$ is evidently a complex line tangent to both surfaces as well as to the integral curve on each surface passing through its points of contact respectively.. The reciprocation may be looked upon as a transformation which takes one point on a complex line over into a second point on the same line.

In the plane, there are two differential equations D and $\bar{D}$ corresponding to S and $\bar{S}$ respectively. Corresponding to the line $P\bar{P}$ is a circle which osculates an integral curve of D and of $\bar{D}$ on the line elements e and $\bar{e}$ respectively. That is, in the plane we have a transformation which takes each line element over into a second line element on the same circle.

Let the equation of the surface S be

$$F(X, Y, Z) = 0. \tag{19}$$

The equations of transformation in space are

$$\left.\begin{aligned} \bar{X} &= \frac{F_Y}{F_Z}, \\[1mm] \bar{Y} &= -\frac{F_X}{F_Z}, \\[1mm] \bar{Z} &= Z + \frac{X F_X}{F_Z} + \frac{Y F_Y}{F_Z}, \end{aligned}\right\} \tag{20}$$

and the equation of the reciprocal surface $\bar{S}$ is found by eliminating X, Y and Z from equations (19) and (20).

Let D have the form

$$f(x, y, p) = 0. \tag{21}$$

The equations of transformation in the plane are

$$
\left.
\begin{aligned}
\bar{x} &= x + \frac{2(1+p^2)f_p f_y}{f_x^2 + f_y^2}, \\[2mm]
\bar{y} &= y - \frac{2(1+p^2)f_p f_x}{f_x^2 + f_y^2}, \\[2mm]
\bar{p} &= \frac{p(f_y^2 - f_x^2) - 2f_x f_y}{(f_y^2 - f_x^2) - 2p f_x f_y},
\end{aligned}
\right\} \quad (22)*
$$

---

*From the Lie transformation

$$
\left.
\begin{aligned}
x &= i\left(Z + \frac{X}{Y}\right), \\[2mm]
y &= Z - \frac{X}{Y}, \\[2mm]
p &= i\,\frac{(1 - Y^2)}{1 + Y^2},
\end{aligned}
\right\} \quad (23)
$$

we obtain the following relations:

$$
\left.
\begin{aligned}
Z &= \frac{x + iy}{2i} \\[2mm]
\frac{X}{Y} &= \frac{x - iy}{2i}, \\[2mm]
Y^2 &= \frac{i - p}{i + p}, \\[2mm]
\frac{4Y^2}{(1 + Y^2)^2} &= 1 + p^2.
\end{aligned}
\right\} \quad (24)
$$

Now, from equations (20) we obtain

$$
\left.
\begin{aligned}
\bar{Z} &= Z + \frac{X F_x}{F_z} + \frac{Y F_y}{F_z}, \\[2mm]
\frac{\bar{X}}{\bar{Y}} &= -\frac{F_y}{F_x}, \\[2mm]
\bar{Y}^2 &= \frac{F_x}{F_y}.
\end{aligned}
\right\} \quad (25)
$$

Assuming $F(X, Y, Z) = f(x, y, p)$, we obtain, by reference to (23),

$$
\left.
\begin{aligned}
F_X &= f_x\,\frac{\partial x}{\partial X} + f_y\,\frac{\partial y}{\partial X} + f_p\,\frac{\partial p}{\partial X} = \frac{1}{Y}(if_x - f_y), \\[2mm]
F_Y &= \frac{X}{Y^2}(f_x - if_y) - if_p\left[\frac{4Y}{(1 + Y^2)^2}\right], \\[2mm]
F_Z &= if_x + f_y.
\end{aligned}
\right\} \quad (26)
$$

Equations (25) then, by reference to (24) and (26), become

$$
\left.
\begin{aligned}
\frac{\bar{x} + i\bar{y}}{2i} &= \frac{x + iy}{2i} - \frac{i(1 + p^2)f_p}{if_x + f_y}, \\[2mm]
\frac{\bar{x} - i\bar{y}}{2i} &= \frac{x - iy}{2i} + \frac{i(1 + p^2)f_p}{if_x - f_y}, \\[2mm]
\frac{i - \bar{p}}{i + \bar{p}} &= \frac{(if_x - f_y)^2(i + p)}{(if_x + f_y)^2(i - p)}.
\end{aligned}
\right\} \quad (27)
$$

Adding and subtracting the first two equations and solving the last for $\bar{p}$ we obtain equations (22).

and the *reciprocal differential equation* is found by eliminating x, y and p from equations (21) and (22).

Let us collect here some of the properties of a general surface and its reciprocal.

(a) The ∞^2 complex tangents to one are also the ∞^2 complex tangents to the other; or in other words, the complex curves on both surfaces have the same ∞^2 tangents. (See Th. 13.)

(b) The P lines tangent to one are also tangent to the other; that is, the P curves of both surfaces lie on the same P scroll. (See Th. 24.)

(c) The L lines tangent to one are also tangent to the other; that is, the L curves of both surfaces lie on the same L scroll. (See Th. 25.)

(d) The complex nodes on one surface are also complex nodes on the other, and the two surfaces touch at these points. (See Th. 27.)

(e) If one surface possesses a singular complex-curve, the other possesses the same singular complex curve; the two surfaces touch along this curve and the complex curves on one surface are tangent one by one to the complex curves on the other along this curve. (See Th. 11.)

The interpretation in the plane is as follows:

THEOREM A: *Associated with a general ordinary differential equation of the first order is a reciprocal differential equation such that:*

(a)* *The ∞^2 osculating circles to the integral curves of one osculate also the integral curves of the other.*

(b) *Both sets of integral curves have the same locus of cusps.*

(c) *The inflectional tangents to both sets of integral curves have the same envelope.*

(d) *This envelope touches the locus of cusps, and the points of contact are common tac-points on both sets of integral curves.*

(e) *If one differential equation has a singular solution, the other has the same singular solution, and the integral curves of one osculate those of the other along this common envelope.*

Consider a general scroll in space. Its reciprocal surface is a second scroll, whose generators are the conjugate polars of the generators of the first, such that:

(f) The complex lines tangent to one scroll along a generator are tangent

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* Each property in the plane follows from the property in space with the corresponding letter given above, where the theorems, which enable us to make the interpretation, are also referred to.

to the reciprocal scroll also along a generator which is the conjugate polar of the first. (See Ths. 2, 3, 21; also Type VIII.)

(g) The complex generators on one scroll are also complex generators on the other. (See Ths. 4, 17.)

THEOREM B: *Associated with every scroll differential equation is a second reciprocal scroll differential equation, and in addition to having the properties* (a), (b), (c) *and* (d).* *stated in Theorem A, they are so related that:*

(f) *Their* ∞^2 *common osculating circles can be grouped into* ∞^1 *families. The circles of each family touch two other circles and osculate the integral curve of both differential equations along the same circle. The tangents to both sets of integral curves along this circle envelope a common concentric circle.*

(g) *The two differential equations have in common a number of integral curves which are circles, and an infinite number of integral curves of both differential equations touch these circles at two points.*

Consider a developable surface in space. Since it has but a single infinity of tangent planes, its reciprocal surface must degenerate into a locus of ∞^1 points — hence a curve. We shall say, then, that such surfaces have degenerate reciprocals. It is evident that the reciprocals of the corresponding differential equations in the plane are then line-element loci of ∞^1 elements. We shall say that such differential equations have degenerate reciprocals.† This fact does not, however, detract from the interest in such equations, but rather adds, for the line-element locus of ∞^1 elements has very important relations to the integral curves.

Consider in particular a general torse:

(h) Its reciprocal is a *skew* curve which is met by all the ∞^2 complex lines tangent to the surface. (See Type I.) Hence

(i) The P curve on a torse and the reciprocal skew curve of the latter lie upon the same P scroll. (See Th. 24.)

(j) The L curve on a torse and the reciprocal skew curve of the latter lie upon the same L scroll. (See Th. 25.)

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* Property (e) is ruled out by Theorem 16.

† The analytic condition that the reciprocal of a differential equation $f(x, y, p) = 0$ be degenerate is expressed by the vanishing of the functional determinant in the equations of transformation (22).

(l) Since the tangent plane touches a torse along a generator, it follows that the complex lines lying in that plane are tangent to the complex curves of the torse along a generator. (See Ths. 10, 2 and 11.)

(k) Since each generator meets the cuspidal edge of the torse, and since the complex curve at that point has a cusp, it follows that one complex line in each plane is a stationary tangent to a complex curve. (See Th. 13.)

(m) The generators of the torse (*i. e.*, the tangents to its cuspidal edge) are the conjugate polar lines of the tangents to its reciprocal skew curve. (See Ths. 2, 3, 8.)

The interpretation in the plane follows:

THEOREM C: *A torse differential equation has the following properties:*

(h) *The ∞^2 osculating circles of its integral curves can be grouped into ∞^1 pencils of tangent circles, the common line element in each pencil giving rise to a skew line-element locus.*

(i) *The point locus of this line-element locus is the locus of cusps on the integral curves.*

(j) *The line locus is the envelope of inflectional tangents to the integral curves.*

(k) *There is one circle of each pencil which hyperosculates an integral curve, thus giving rise to a locus of hyperosculation.**

(l) *The circles of each pencil osculate the integral curves along a circle, and the tangents to the integral curves along the latter envelope a concentric circle.*

(m) *The first set of circles thus determined are enveloped by the cuspidal locus and the locus of hyperosculation, while the concentric circles touch both the envelope of inflectional tangents and the envelope of tangents drawn along the locus of hyperosculation.*

We associate with every general torse in space a second torse defined by the envelope of the ∞^1 complex planes of the points on the cuspidal edge of the first. We shall refer to these as *related* torses. They have the following properties:

(n) The cuspidal edge of one is the reciprocal skew curve of the other.

(o) The complex lines tangent to one along its cuspidal edge are tangent also to the other along its cuspidal edge; that is, the ∞^1 stationary tangents to

---

* The second line-element locus of ∞^1 elements thus determined is that corresponding to the cuspidal edge in space.

the complex curves on one torse are also stationary tangents to the complex curves on the other. (See Th. 13.)

(p) Consider one of these common stationary tangents. Let it meet the cuspidal edge of the first torse at P and the cuspidal edge of the related torse at P'. Then the complex lines through P touch the related torse along that generator which is tangent to its cuspidal edge at P', and the complex lines through P' touch the other torse along that generator which is tangent to its cuspidal edge at P. These two generators are conjugate polar lines. (See Ths. 2, 3 and 8.)

The interpretation follows:

THEOREM D : *Associated with every torse differential equation is a related* * *torse differential equation such that:*

(n) *The cuspidal locus of the integral curves of one is the locus of hyperosculation on the integral curves of the other, and the envelope of inflectional tangents to the integral curves of one is the envelope of the tangents drawn to the integral curves along the locus of hyperosculation of the other.*

(o) *The ∞^1 circles which hyperosculate the integral curves of one hyperosculate also the integral curves of the other.*

(p) *Let C be one of these common hyperosculating circles and let P and P' be the two points of hyperosculation. Consider the two pencils of circles tangent to C at P and P' respectively. The circles of one pencil osculate the integral curves of one differential equation, and the circles of the second pencil osculate the integral curves of the other along the same circle C_1. The tangents drawn to both sets of integral curves along C_1 envelope a common concentric circle C_2. The ∞^1 circles C_1 are enveloped by both cuspidal loci, while the concentric circles C_2 touch both envelopes of inflectional tangents.*

It is interesting to note that the entire configuration discussed in the last theorem is built upon a single skew† line-element locus of ∞^1 elements, just as the configuration of related developables in space is built upon a single skew curve — the cuspidal edge of either one of them. Since one cuspidal edge

---

* Not reciprocal.

† It is essential that the line-element locus be skew and not plane. (See Theorem E.)

uniquely determines the other, so in the plane we associate a second skew line-element locus with the first. The two sets of pencils of tangent circles determined by these line elements are respectively the osculating circles to the integral curves of the related torse differential equations discussed in Theorem D.

Consider now a cone:

(q) Its reciprocal is a *plane* curve which is met by all the ∞^2 complex lines tangent to the surface. These include also those complex lines lying in the complex plane of the vertex. (See Type II.)

See (i) and (j) under torse.

(r) The complex lines lying in a tangent plane are tangent to the complex curves on the cone along an element. (See Ths. 10, 2 and 11.)

(s) The elements of the cone all pass through its vertex (see Th. 7) and are the reciprocal polars of the tangents to its reciprocal plane curve. (See Ths. 2, 3 and 8.)

The interpretation is as follows:

THEOREM E: *A cone differential equation has the following properties:*

(q) *The ∞^2 osculating circles of its integral curves can be grouped into ∞^1 pencils of tangent circles. One circle from each pencil belongs to a special pencil. The common line elements of these pencils thus give rise to a plane line-element locus and a particular isolated line element.*[*]

See (i) and (j) of Theorem C.

(r) *The circles in each general pencil osculate the integral curves along a circle, and the tangents to the integral curves along the latter envelope a concentric circle.*

(s) *The first set of circles thus determined all touch the cuspidal locus and pass through the point of the particular line element, while the concentric circles all touch the envelope of inflectional tangents and the line of the particular line element.*

Up to this point we have considered the reciprocals of general surfaces, scrolls, torses and cones together with the corresponding interpretation in the plane. This has also involved the reciprocation of non-complex curves in space. The interpretation of complex ruled surfaces and their reciprocals gives rise to

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[*] Evidently the line element corresponding to the vertex of the cone.

nothing of particular interest in the plane from the standpoint of the ordinary differential equation of the first order, for the osculating circles in this case reduce to the integral curves themselves. However, the reciprocation of all the distinct types of surfaces and curves will be found to be equally important in the discussion of complex congruences and their interpretation which will appear in a later paper.

REMARK. It is evident that the geometry within a linear line complex can be equally well interpreted by other transformations aside from the one used in this paper. (See "Geometrie der Berührungstransformationen," p. 238.) But this one is perhaps the most important, since it contributes to the theory of the osculating circle.

SHEFFIELD SCIENTIFIC SCHOOL, YALE UNIVERSITY, May, 1910.

Binary Modular Groups and their Invariants.

By Leonard Eugene Dickson.

1. In the first part of this paper I determine all subgroups of the group Γ composed of all binary transformations of determinant unity with coefficients in the Galois field F of order p^n. The order of Γ is

$$\omega = p^n(p^{2n} - 1).$$

I determined the subgroups of Γ in the spring of 1904 and made use of the results in investigating [*] the subgroups of the general ternary and quaternary linear groups modulo p, as well as in my study of finite algebras.[†]

The subgroups of Γ may be derived (as in § 9) from the subgroups of the linear fractional group. We may however proceed independently (§§ 2-7). The latter method naturally brings out more clearly the properties of the homogeneous groups, and moreover furnishes material needed in the construction of the invariants (§§ 10-13). The linear fractional groups may be derived by inspection from the homogeneous groups.

The exceptional character of the case $p = 2$ is more marked in the case of homogeneous groups than in the case of fractional groups. Moreover, the homogeneous and fractional groups are identical if $p = 2$. For these reasons I assume here that $p > 2$.

Canonical Forms and Conjugacies of the Transformations of Γ.

2. Each transformation of Γ is given either of the notations

$$\begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}: \quad \begin{aligned} x' &= \alpha x + \beta y \\ y' &= \gamma x + \delta y \end{aligned} \quad (\alpha\delta - \beta\gamma = 1). \quad (1)$$

If this replaces a linear function l by ρl, then ρ is a root of the characteristic equation

$$\Delta(\rho) = \rho^2 - (\alpha + \delta)\rho + 1 = 0. \quad (2)$$

[*] American Journal of Mathematics, Vol. XXVII (1905), Vol. XXVIII (1906).
[†] Göttingen Nachrichten, 1905, pp. 358-393; see § 4.

If $\Delta(\rho)$ is irreducible in F, (1) has the canonical form

$$T_\kappa = \begin{pmatrix} \varkappa & 0 \\ 0 & \varkappa^{-1} \end{pmatrix}, \quad \varkappa^{p^n+1} = 1, \quad \varkappa^2 \neq 1. \tag{3}$$

If $\Delta(\mathfrak{z})$ has two distinct roots in F, (1) is conjugate within Γ with

$$T_\lambda = \begin{pmatrix} \lambda & 0 \\ 0 & \lambda^{-1} \end{pmatrix}, \quad \lambda^{p^n-1} = 1, \quad \lambda^2 \neq 1. \tag{4}$$

But if the roots are equal, (1) is conjugate within Γ with

$$S_{\pm 1, \beta} = \begin{pmatrix} \pm 1 & \beta \\ 0 & \pm 1 \end{pmatrix}. \tag{5}$$

Now T_λ transforms $S_{\pm 1, \beta}$ into $S_{\pm 1, \lambda^2\beta}$. Thus the transformations (5) with $\beta \neq 0$ are conjugate with $S_{\pm 1, 1}$ or $S_{\pm 1, \nu}$, where ν is a fixed not-square. The latter types are seen to be not conjugate.

Commutative and Di-cyclic Subgroups.

3. If λ is a primitive root of F, T_λ generates a cyclic C_{s-1}, where $s = p^n$. Here $s > 3$, in view of the assumption (4) that $\lambda^2 \neq 1$. Then the only transformations (1) commutative with T_λ are the T_a, and the only ones transforming T_λ into its inverse $T^{\lambda-1}$ are the $T_a T$, where $T = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$. Evidently T_λ and T^{-1} are the only transformations of C_{s-1} conjugate with T_λ. Hence C_{s-1} is invariant only in a di-cyclic $G_{2(s-1)}$.

A di-cyclic G_{4k} is generated by two operators A and B, where A is of period $2k$ and $B^2 = A^k$, $B^{-1}AB = A^{-1}$; it is said to have the cyclic base $C_{2k} = \{A\}$. Two operators BA^i and BA^j $(i < 2k, \ j < 2k)$ are conjugate within G_{4k} if and only if i and j are both even or both odd. Since the inverse of BA^i is BA^{i+k}, the cyclic C_4 generated by the BA^i form one or two conjugate sets according as k is odd or even. Let d be any divisor > 1 of k and set $\delta = k/d$. If μ is a fixed one of the integers $0, 1, \ldots, \delta - 1$, BA^μ extends the cyclic base $\{A^\delta\}$ of order $2d$ to a di-cyclic $G_{4d}^{(\mu)}$. These δ groups are all conjugate within G_{4k} if δ is odd, or if δ is even and k odd; but fall into two distinct sets of conjugates if δ and k are both even. If $d \neq 2$, this process yields every di-cyclic subgroup of G_{4k}, since a G_{4d} has a single cyclic C_{2d}. If $d = 2$, then k is even and we may set $k > 2$. The only operators of period 4 in G_{4k} are

$$A^{\pm k/2}, \quad BA^i \qquad (i = 0, 1, \ldots, 2k - 1).$$

Hence a di-cyclic subgroup G_8 contains at least four BA^i and hence two distinct operators BA^r and BA^s, not inverse to each other. Thus

$$r \not\equiv s, \quad r \not\equiv s + k \pmod{2k}.$$

Hence G_8 contains their product A^{s-r+k}, which is neither A^k nor the identity. Hence G_8 contains $A^{k/2}$ and may be based on the cyclic $\{A^{k/2}\}$. *Every di-cyclic subgroup of G_{4k} may be based on a cyclic $\{A^\delta\}$, where $\delta = k/d$ is a divisor of k. For each divisor δ, there are δ di-cyclic subgroups $G_{4d} = \{A^\delta, BA^\mu\}$, $\mu = 0, 1, \ldots, \delta - 1$, forming one system or two systems of conjugate subgroups according as δ and k are not both even or both even.*

In the $GF[p^{2n}]$, $x^{p^m+1} = 1$ has a primitive root x. Then T_κ generates a cyclic C_{s+1}, invariant only in a di-cyclic $G_{2(s+1)}$.

We next determine the di-cyclic subgroups of Γ whose cyclic bases are subgroups of $C_{s\mp1}$. Now Γ contains $\frac{1}{2}s(s \pm 1)$ conjugate cyclic $C_{s\mp1}$, each invariant only in a di-cyclic $G_{2(s\mp1)}$. The latter are all conjugate under Γ, since an operator which transforms G into G' transforms every operator commutative with G into an operator commutative with G'. If $2d_\mp$ is any even divisor > 2 of $s \mp 1$ and $\delta_\mp$ is the quotient, Γ contains $\frac{1}{2}s(s \pm 1)$ conjugate cyclic $C_{2d_\mp}$, each serving as a cyclic base for $\delta_\mp$ di-cyclic $G_{4d_\mp}$, forming one system or two systems of conjugates under $G_{2(s\mp1)}$ according as not both or both $\delta_\mp$ and $\frac{1}{2}(s \mp 1)$ are even (by above theorem for $k = \frac{1}{2}(s \mp 1)$). For $d_\mp \neq 2$, two subgroups $G_{4d_\mp}$ of $G_{2(s\mp1)}$ are conjugate within the latter if conjugate within Γ. Indeed, the transforming operator must be commutative with $C_{2d_\mp}$, the only cyclic subgroup of this order in either of the $G_{4d_\mp}$, and hence with the unique cyclic $C_{s\mp1}$ containing it. *Hence if $2d_\mp$ is any even divisor > 4 of $s \mp 1$ and the quotient is $\delta_\mp$, Γ contains in all $s(s^2 - 1) \div 4d_\mp$ di-cyclic $G_{4d_\mp}$, forming one system or two systems of conjugates according as $\delta_\mp$ and $\frac{1}{2}(s \mp 1)$ are not both even or both even. In the first case a $G_{4d_\mp}$ is invariant only under itself; in the second case, under a di-cyclic $G_{8d_\mp}$.*

Consider next the divisor $2d_\mp = 4$ of $s \mp 1$. The sign $\mp$ must be such that $\frac{1}{4}(s \mp 1)$ is an integer σ. All the transformations of period 4 of Γ belong to the conjugate cyclic $C_{4\sigma}$. Each di-cyclic G_8 contains 3 cyclic C_4. Now Γ contains $\frac{1}{2}s(s \pm 1)$ conjugate C_4, each serving as a base for δ di-cyclic G_8. Hence Γ contains in all $\frac{1}{24}s(s^2 - 1)$ di-cyclic G_8.

A maximum di-cyclic $G_{8\sigma}$ contains σ di-cyclic G_8, forming one system or two systems according as σ is odd or even; namely, according as $s = p^n$ is of the

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form $8h \pm 3$ or $8h \pm 1$. Since the $G_{8\sigma}$ are all conjugate under Γ, it follows that, if σ is odd, all the G_8 are conjugate; while if σ is even, they form at most two systems of conjugates under Γ. Suppose that, for σ even, they form a single system. Then in view of the total number of G_8, each would be invariant under exactly 24 transformations of a subgroup G_{24}. But, for σ even, each G_8 is one of $\sigma/2$ conjugates under a certain $G_{8\sigma}$ and is therefore invariant under a subgroup of order 16 of $G_{8\sigma}$. Hence the $\frac{1}{24} s(s^2 - 1)$ di-cyclic G_8 contained in Γ form one system or two systems of conjugates according as $s = p^n$ has the form $8h \pm 3$ or $8h \pm 1$. In the former case, a G_8 is invariant in exactly a G_{24}; in the latter case, under a G_{48}.

For $\beta \neq 0$, $S_{1,\beta}$ is of period p and $S_{-1,\beta}$ of period $2p$. Now

$$S_{a,b} = \begin{pmatrix} a & b \\ 0 & a^{-1} \end{pmatrix},\tag{6}$$

whose inverse is $S_{a^{-1},-b}$, transforms $S_{\pm1,\beta}$ into $S_{\pm1,\tau}$, where $\tau = a^2\beta$, while no transformation other than the $S_{a,b}$ transforms $S_{\pm1,\beta}$ into one of like type. The $2s$ transformations $S_{\pm1,\beta}$, where β ranges over the field, form a commutative group, since

$$S_{\pm1,\beta} S_{\pm1,\delta} = S_{1,\pm\beta\pm\delta}, \quad S_{-1,\beta} S_{1,\delta} = S_{1,\delta} S_{-1,\beta} = S_{-1,\beta-\delta}.\tag{7}$$

This commutative group G_{2s} is therefore invariant only under the group $G_{s(s-1)}$ of the transformations (6), and hence is one of $s + 1$ conjugates under Γ. The same is true of the commutative group G_s formed by the $S_{1,\beta}$.

By *Linear Groups*, § 249, with (2; 1) replaced by 1, this G_s has exactly

$$\frac{(p^n - 1)(p^n - p)(p^n - p^2)\cdots(p^n - p^{m-1})}{(p^m - 1)(p^m - p)(p^m - p^2)\cdots(p^m - p^{m-1})}\tag{8}$$

subgroups G_{p^m} and each is invariant in a largest group H of order $lp^n(p^k - 1)$, where $l = 2$ or 1 according as n/k is even or odd, while the value of k depends upon the particular G_{p^m} chosen. Thus the G_{p^m} is one of a system of $(p^{2n} - 1) \div l(p^k - 1)$ conjugates under Γ.

Consider next a subgroup of G_{2s} containing $S_{-1,\beta}$. If $\beta \neq 0$, it contains $S_{-1,\beta}^2 = S_{1,-2\beta}$, by (7), and hence $S_{1,c\beta}$, where c is any integer. Hence it contains $S_{-1,\beta} S_{1,\beta} = S_{-1,0}$. Thus in every case the subgroup contains $S_{-1,0} = T_{-1}$, and is therefore a G_{2p^m} given by the extension of one of the preceding G_{p^m} by T_{-1}. This G_{2p^m} contains a single G_{p^m}, while T_{-1} is invariant under Γ. Hence, if $m > 0$, G_{2p^m} is invariant only under the above group H.

Non-commutative Subgroups of $G_{s(s-1)}$.

4. This group G is composed of the transformations (6); viz., $S_{1,\mu} T_a$, where $\mu = b/a$. A rectangular array for G may be formed by taking as the first row the transformations $S_{1,\mu}$, which form the invariant subgroup G_s, and as right-hand multipliers the T_a of the cyclic C_{s-1}. In any subgroup G' of G the totality* of transformations of period p give rise to a commutative G_{p^m} invariant in G'. A rectangular array for G' with the transformations of G_{p^m} in the first row has the property that the transformations in each row are all found in a row of the array for G. In fact, two transformations A and B of G' lie in the same row or in different rows of the array for G' according as AB^{-1} is or is not in G_{p^m}; namely, is or is not in the first row of the array for G. Hence the quotient-group G'/G_{p^m} is a subgroup G_d of the cyclic group G/G_{p^n}.

For a $a^2 \neq 1$, the period of $S_{a,b}$ is the exponent to which a belongs, since

$$S_{a,b}^k = S_{a^k, bc}, \quad c = a^{k-1} + a^{k-3} + \dots + a^{-k+1} = a^{-k+1}\left(\frac{a^{2k}-1}{a^2-1}\right).$$

Hence G contains $2(s-1)$ transformations $S_{\mp 1,\beta}$ of period p or $2p$, and $s^2 - 3s + 2$ of period dividing $s-1$. Hence G contains s cyclic $C_{s-1}^{(b)}$, no two having in common an operator other than $T_{\pm 1}$. They are conjugate within G, since $S_{1,\mu}$ transforms $S_{a,b}$ into $S_{a,B}$, where $B = b + \mu(a^{-1} - a)$. Their subgroups $G_d^{(b)}$, for the various divisors d of $s-1$, furnish all the cyclic subgroups of G other than those of period p or $2p$. We proceed as in *Linear Groups*,† p. 271, beginning with line 22, and replacing G_{p^m} by G_{2p^m} (composed of the $S_{\pm 1,\beta}$) in the last line. We conclude that G' is one of $p^{n-m}(p^{2n}-1) \div l(p^k-1)$ conjugates under Γ. Here k and l have the same meaning as in § 3.

Remaining Subgroups Containing Operators of Period p.

5. We proceed as in *Linear Groups*, pp. 272–278, with the following changes.‡ In place of lines 7 and 8 on p. 273, read: "there are d marks η, the distinct powers of a primitive root of $\eta_0^d = +1$." In equations (251) and (253), replace ± 2 by $+2$. At the bottom of p. 273 and on p. 274, replace (2; 1) by 1.

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* If no operator of period p occurs, G' is a cyclic subgroup of one of the C_{s-1} and has been listed in § 3.

† In lines 3 and 11 from bottom, change ∞, 0 to ∞, λ, and "within which G_{p^m} is self-conjugate" to "which transforms G_{p^m} into itself."

‡ Errata on p. 274: l. 8, interchange k and m; l. 14, delete "with n/k odd."

Thus (252) now reads

$$p^m - 1 \leq d \leq l(p^k - 1) \leq l(p^m - 1),$$

whence $k = m$. But d divides $l(p^k - 1)$. Hence either (A) $d = p^k - 1$ or (B) $l = 2$ and $d = 2(p^k - 1)$. On p. 275, line 6, we employ $x = -1$ (instead of $+1$) and reach the desired result. For $p^k = 3$, the treatment requires the following modification. Since the subgroup contains P_η, η any mark $\neq 0$ in the $GF[p^k]$, it contains the $p^m d = 6$ transformations $V_{1,\lambda} P_{\pm 1}$, $\lambda = 0, 1, -1$. The $\alpha + \delta$ of $V_j' = V_{1,\pm 1} V_j$ is $\alpha_j + \delta_j \pm \gamma_j$, which may be made zero by choice of $\gamma_j = \pm 1$. Employing $P_{\gamma_j} V_j'$, we have the new γ_j unity and $\alpha + \delta$ still zero. It follows that, for $p > 2$, the group in case (A) is the total group B_k of transformations of determinant 1 in the $GF[p^k]$.

For use in (B), where $p > 2$, we replace in the lemma on p. 274 period 2 by period 4, namely, $V_j^2 = P^{-1}$. In § 253, there are now d marks η; the orders of the groups are now twice as great; the dihedron is now di-cyclic. Instead of T, we consider the C_4 generated by T_0. In the third line of p. 277, read " $2fp^k(p^k - 1)$ substitutions of period 4 "; in l. 11 read: "distinct from V_j and $V_j P^{-1}$, and of period 4." We thus reach $2(p^k - 1)$ substitutions $V_{\eta,\lambda} V_j$ of period 4, and hence $p^k - 1$ cyclic C_4. If M is the number of the V_j leading to a single $C_4 = (V_{\eta,\lambda} V_j)$, the total number of the latter is given in the text. It follows that either $\Omega = 2p^k(p^{2k} - 1)$ or else $\Omega = 120$ and $p^k = 3$. In the first case we employ the subgroup of the $p^k(p^k - 1)$ transformations $V_{\kappa,\lambda}$ (of index 2 under the group of all the $V_{\eta,\lambda}$) and show that it is extended by the V_j' to the group B_k of all binary transformations of determinant 1 in the $GF[p^k]$. Hence $G_\Omega = \{B_k, P_{\eta_0}^2\}$, where $P_{\eta_0}^2$ belongs to B_k, and η_0 is the square root of a primitive root of the $GF[p^k]$. Thus G_Ω is a group in the $GF[p^{2k}]$.

In the second case, G_{120} has one set of $1 + fp^k = 10$ conjugate C_3, and one set of 15 conjugate C_4 each invariant in exactly a di-cyclic G_8. Hence there are 5 conjugate G_8. It is shown in §§ 6, 7 that G is of the homogeneous icosahedral type and occurs as a subgroup of Γ in the $GF[3^n]$, n even.

As in § 255, the largest subgroup of Γ in which the total binary B_k in the $GF[p^k]$ is invariant is B_k if n/k is odd, and $\{B_k, P_{\eta_0}\}$ if n/k is even; while the latter is invariant only under itself. The groups of the latter type (occurring only when n/k is even) form 2 systems of conjugates under Γ. The groups of type B_k form two systems of conjugates if n/k is even, and one system if n/k is odd.

Subgroups Containing no Operator of Period p.

6. Every transformation other than $T_{\pm 1}$ of such a subgroup G_Ω lies in a unique largest cyclic subgroup C_d of G_Ω. Two such C_d have in common no operator other than $T_{\pm 1}$. According as C_d is invariant within G_Ω only under itself or under a di-cyclic* G_{2d} based on C_d, it is one of a system of Ω/d or $\Omega/2d$ conjugates under G_Ω. Let r be the number of such systems. The enumeration of the transformations of G_Ω leads to the relations

$$\Omega = \delta + \sum_{i=1}^{r} (d_i - \delta) \frac{\Omega}{t_i d_i} \qquad (f_i = 1 \text{ or } 2), \qquad (9)$$

$$\Omega \geq f_i d_i \qquad\qquad (i = 1, \ldots, r), \qquad (10)$$

where $\delta = 2$ if G contains T_{-1}, $\delta = 1$ if it does not. Indeed, if G contains T_{-1}, every G_{d_i} contains T_{-1}. Since T_{-1} is the only transformation of period 2, it suffices to show that d_i is even. If S is of odd period σ, $S T_{-1}$ is of period 2σ and $(S T_{-1})^{\sigma-1} = S^{-1}$, so that the cyclic $\{S T_{-1}\}$ contains S. Next, if G does not contain T_{-1}, Ω and each d_i are odd. In each case Ω and d_i are multiples of δ and we may set

$$\Omega = \Omega' \delta, \quad d_i = d_i' \delta, \qquad (i = 1, \ldots, r).$$

When these values are inserted in (9) and (10), we obtain the relations, written in accented letters, at the beginning of § 256 of *Linear Groups*. Employing the results obtained, we reach the following conclusions. If $r = 1$, then $f_1 = 1$, $\Omega' = d_1'$, and G is a cyclic C_{d_1}. If $r = 2$, we may interchange f_1 and f_2 if necessary and set $f_1 = 1$, $f_2 = 2$. Either $d_1' = 2$, $\Omega' = 2d_2'$, or $d_1' = 3$, $d_2' = 2$, $\Omega' = 12$. In each case, Ω is even and G contains T_{-1}, so that $\delta = 2$. In the first case, $d_1 = 4$, $d_2 = 2d_2'$, where d_2' is odd (otherwise C_{d_1} would not be maximal), and G is a di-cyclic G_{2d_2}, already considered (§ 3). In the second case, $d_1 = 6$, $d_2 = 4$, $\Omega = 24$. Thus G_{24} contains a system of 4 cyclic C_6 each invariant only under itself. Since they have only $T_{\pm 1}$ in common, G_{24} is isomorphic with a subgroup $G_{12}^{(t)}$, necessarily the alternating group on 4 letters. Since the latter has an invariant G_4, G_{24} has an invariant G_8. This also follows from the fact that the 4 C_4 contain 8 operators of period 6, 8 of period 3, so that there are at most 8 operators of periods powers of 2; but G contains a di-cyclic G_8 based

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*Dihedron in the case $p = 2$ not considered here; then $\delta = 1$ below.

on C_{d_2}. Hence the di-cyclic G_8 is invariant. Thus* G_{24} is of the homogeneous tetrahedral† type, with the generational relations

$$A^4 = I, \quad B^2 = A^2, \quad B^{-1}AB = A^{-1}, \quad C^3 = I, \quad C^{-1}AC = B, \quad C^{-1}BC = AB. \quad (11)$$

Although falling under another heading, we note that, for $p^n = 3$, the total group Γ is of this type‡ (cf. § 3).

For $r = 3$, each $f_i = 2$ and we may set $d_3' = 2$, whence $\delta = 2$. Either $d_2' = 2$, $\Omega' = 2 d_1'$, whence G_Ω is a ◆-cyclic G_{2d_1}, or $d_2' = 3$, $d_1' = 3, 4, 5$, $\Omega' = 12, 24, 60$, respectively. For $d_1' = 3$, we have $d_1 = d_2 = 6$, $d_3 = 4$, $\Omega = 24$; this case is to be excluded, since C_{d_1} is invariant only under a $G_{f_1 d_1} = G_{12}$ and hence is one of two conjugates, whereas the operators of C_{d_2} transform it into at least 3 distinct groups. For $d_1' = 4$, we have $d_1 = 8$, $d_2 = 6$, $d_3 = 4$, $\Omega = 48$. Since $f_i = 2$, there are 4 conjugate C_6, each invariant in a di-cyclic G_{12}. No two of the latter have a C_3 in common. Hence the group common to all four G_{12} is a C_4 or is composed of $T_{\pm 1}$. The first case is excluded since a C_4 is not invariant in G_{48}. Hence $T_{\pm 1}$ alone transform each of the 4 conjugate C_6 into itself. Thus G_{48} is isomorphic with the symmetric group on 4 letters, having an invariant G_4. Hence G_{48} contains an invariant di-cyclic G_8. Hence (§ 3) this G_{48} occurs only when $p^n = 8 h \pm 1$ and is then uniquely determined by its invariant G_8. We may determine G_{48} abstractly by the properties that it contains a single operator A of period 2 and that the quotient-group $G_{48}/\{I, A\}$ is of the octahedral type. The latter is generated by B and C, where $B^4 = C^3 = I$, $(BC)^2 = I$. Arrange the operators of G_{48} into 24 sets S_i, $S_i A$. It must be possible to choose two sets S_1, $S_1 A$ and S_2, $S_2 A$ such that

$$S_1^4, \quad (S_1 A)^4, \quad S_2^3, \quad (S_2 A)^3, \quad (S_1 A^i \cdot S_2 A^j)^2$$

are all in the set I, IA. If $S_2^3 = I$, then $(S_2 A)^3 = A$; if $S_2^3 = A$, then $(S_2 A)^3 = I$. Hence, by choice of the notation, we may set $S_2^3 = I$. Since $S_1 S_2 \neq A, I$, we have $(S_1 S_2)^2 \neq I$. Hence $(S_1 S_2)^2 = A$. If $S_1^4 = I$, then $S_1^2 = A$ or I, whereas $B^2 \neq I$. Hence

$$S_1^4 = A, \quad S_2^3 = I, \quad (S_1 S_2)^2 = A, \quad A^2 = I, \quad AS_1 = S_1 A, \quad AS_2 = S_2 A. \quad (12)$$

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* It is not the direct product of G_8 and a C_3, and hence by Burnside, *Theory of Groups*, p. 103, case (iv), is of type (11). Note that Burnside's proof is faulty; $C^{-3}AC^3$ is A and not A^{-1}. But if $C^{-1}BC = B^{-1}A^{-1}$, set $A_1 = AB$, $B_1 = A^{-1}$. Then $C^{-1}A_1C = B_1$, $C^{-1}B_1C = B^{-1} = ABA^{-1} = A_1B_1$, so that his conclusion is proved.

† $A = (iz_1, -iz_2)$, $B = (-z_2, z_1)$, $C = \left(\frac{(i-1)}{2}(z_1+z_2), \frac{(i+1)}{2}(z_1-z_2)\right)$.

‡ $A = \begin{pmatrix} 0 & 1 \\ 2 & 0 \end{pmatrix}$, $B = \begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix}$, $C = \begin{pmatrix} 1 & 0 \\ 2 & 1 \end{pmatrix}$, modulo 3.

This is a complete set of generational relations for a G_{48}. Indeed, every element can be written in the form SA or S, where S is a product of S_1, S_2, and includes 24 distinct operators in view of the relations defining the octahedral G_{24}. This* G_{48} contains a single G_{24} of type (11), 12 operators of period 8, 8 of period 6, 18 of period 4, 8 of period 3, 1 of period 2, and identity. A linear group of this type is given in § 11.

For $d_1' = 5$, we have $d_1 = 10$, $d_2 = 6$, $d_3 = 4$, $\Omega = 120$. Each C_4 is invariant only in a di-cyclic G_8. Hence there are $15/3$ conjugate G_8. Thus each G_8 is invariant only in a G_{24}, necessarily of type (11). The 5 conjugate G_{24} have only $T_{\pm 1}$ in common. Indeed, their common operators form an invariant subgroup of G_{120} and hence of each G_{24}. But a homogeneous tetrahedral G_{24} has besides I, C_2, G_{24} (cases requiring no further discussion) the single further invariant sub-group G_8. But the five G_8 in the five G_{24} are all distinct. Hence $G_{120}/\{T_{\pm 1}\}$ is the alternating group on 5 letters, viz., an icosahedral group. Further, G_{120} has a single operator T_{-1} of period 2. Hence† there is only one type of such a group and its generational relations are

$$A^2 = I, \quad AB = BA, \quad AC = CA, \quad B^3 = I, \quad C^5 = I, \quad (BC)^2 = A. \qquad (13)$$

Although listed elsewhere, the total group Γ for $p^n = 5$ is of this type.‡

Number and Conjugacy of the Homogeneous Icosahedral Subgroups.

7. For $p^n = 5$, Γ itself is such a G_{120}. For $p^n = 5^n$, we employ the result at the end of § 5 and conclude that the G_{120} fall into two systems each of $5^n(5^{2n}-1)/240$ conjugate groups if n is even, but into one system of $5^n(5^{2n}-1)/120$ conjugates if n is odd.

For use below we show that a G_{120} has 120 sets of generators A, B, C. For $p^n = 5$, $\Gamma = G_{120}$ has 6 cyclic C_5, and each operator of period 5 is conjugate with at least one $\begin{pmatrix} 1 & \mu \\ 0 & 1 \end{pmatrix}$, μ not a multiple of 5. Taking the latter to be C, and giving B the form (14), we find that BC has $\alpha' = \alpha + \mu\gamma$, $\delta' = \delta$. Hence $(BC)^2 = A = T_{-1}$ if, and only if, $\alpha + \mu\gamma + \delta = 0$. Thus $\gamma = \mu^{-1}$. Then $\beta = -\mu(1 + \alpha + \alpha^2)$. Thus there are 5 operators B for each C and hence 24.5 sets of generators.

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* It is of type 52 in Miller's list, *Quarterly Journal*, Vol. XXX, p. 258.

† Burnside, *Theory of Groups*, p. 377.

‡ For example, $A = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$, $B = \begin{pmatrix} 0 & -1 \\ 1 & -1 \end{pmatrix}$, $C = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$, modulo 5.

Next, let $p \neq 2$, $p \neq 5$. Then $p^n(p^{2n} - 1)$ is divisible by 120 if, and only if, $p^{2n} - 1$ is divisible by 5. First, let $\lambda = \frac{1}{5}(p^n - 1)$ be an integer. Let ρ be a primitive root of the field; then ρ^λ is of period 5. Set

$$C = \begin{pmatrix} \rho^\lambda & 0 \\ 0 & \rho^{-\lambda} \end{pmatrix}, \quad B = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}, \quad \alpha\delta - \beta\gamma = 1, \quad \alpha + \delta + 1 = 0.^* \qquad (14)$$

Then BC has $\alpha' = \rho^\lambda\alpha$, $\delta' = \rho^{-\lambda}\delta$. Hence $(BC)^2 = A = T_{-1}$ if, and only if,

$$\rho^\lambda\alpha + \rho^{-\lambda}\delta = 0.$$

The three conditions give

$$\alpha = \frac{1}{\rho^{2\lambda} - 1}, \quad \delta = -\rho^{2\lambda}\alpha, \quad \beta\gamma = \frac{-c}{(\rho^{2\lambda} - 1)^2}, \quad c \equiv \rho^{4\lambda} - \rho^{2\lambda} + 1.$$

If $c = 0$, then $\rho^{6\lambda} = -1$, whereas ρ^λ is of period 5. Hence to each of the $p^n - 1$ values $\neq 0$ of β there corresponds a single value of γ. But Γ contains (§ 3) exactly $\frac{1}{2}p^n(p^n + 1)$ conjugate cyclic C_5. Hence there are $2p^n(p^{2n} - 1)$ sets of generators A, B, C of homogeneous icosahedral subgroups.

Next, let $g = \frac{1}{5}(p^n + 1)$ be an integer. Our group Γ is simply isomorphic with the group of binary hyperorthogonal transformations in the $GF[p^{2n}]$:

$$\begin{pmatrix} \alpha & \beta \\ -\bar\beta & \bar\alpha \end{pmatrix}, \quad \alpha\bar\alpha + \beta\bar\beta = 1, \qquad (15)$$

where $\bar\alpha$ denotes α^{p^n}. Let B have the form (15), so that $\alpha + \bar\alpha + 1 = 0$. Set

$$C = \begin{pmatrix} J^g & 0 \\ 0 & \bar{J}^g \end{pmatrix}, \quad J\bar{J} = 1.$$

Now BC is of the form (15) with $\alpha' = \alpha J^g$. Hence $(BC)^2 = T_{-1}$ gives

$$\alpha J^g + \bar\alpha\bar{J}^g = 0.$$

For a given J, these conditions are satisfied if, and only if,

$$\alpha = \frac{\bar{J}^g}{J^g - \bar{J}^g}, \quad \beta\bar\beta = 1 + (J^g - \bar{J}^g)^{-2}.$$

The final sum is an element $\neq 0$ of the $GF[p^n]$, so that there are $p^n + 1$ values of β. Now Γ contains $\frac{1}{2}p^n(p^n - 1)$ conjugate C_5. Hence again there are $2p^n(p^{2n} - 1)$ sets of generators of homogeneous icosahedral subgroups.

But each G_{120} has 120 sets of generators. Hence, *when $p^n \pm 1$ is divisible by 5, Γ contains in all $p^n(p^{2n} - 1)/60$ homogeneous icosahedral groups.*

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* This is the necessary and sufficient condition that $B^3 = I$.

In the first case T_{ρ^e} transforms C into itself and B into a transformation with α and δ unaltered, but with β replaced by $\rho^{2e}\beta$. The latter may be made equal to unity or a particular not-square. Since the C_5 are all conjugate, it follows that there are at most two systems of conjugate G_{120}. But if there were a single system, their number would be at most $p^n(p^{2n}-1)/120$, contrary to the above. Hence[*] *there are two systems of conjugate homogeneous icosahedral groups within* Γ, *and each is invariant only under itself.*

In the second case we employ the transformer T_{J^e} to (15) and find that α is unaltered, while β is multiplied by J^{2e}. But the ratio of two values of β is a power of J. Hence we may set $\beta = 1$ or J. Hence the preceding result holds also in this case.

Summary of the Subgroups of Γ $(p > 2)$.

8. One invariant C_2; $\frac{1}{2}p^n(p^n \pm 1)$ conjugate cyclic $C_{d_{\mp}}$ for every divisor $d_{\mp} > 2$ of $p^n \mp 1$; $p^n(p^{2n}-1) \div 4e_{\mp}$ di-cyclic $G_{4e_{\mp}}$, forming one system or two systems of conjugates according as $(p^n \mp 1)/2e_{\mp}$ and $(p^n \mp 1)/2$ are not both even or both even, where $e_{\mp}$ is any divisor > 2 of $(p^n \mp 1)/2$; $p^n(p^{2n}-1)/24$ di-cyclic G_8, forming one system or two systems of conjugates according as $p^n = 8h \pm 3$ or $p^n = 8h \pm 1$; $N(p^n + 1)$ commutative G_{p^m} each one of $(p^{2n}-1) \div l(p^k-1)$ conjugates, where N is given by (8) and $l = 2$ or 1 according as n/k is even or odd, while k depends upon the particular G_{p^m} $(k = n$ if $m = n)$; $N(p^n + 1)$ commutative G_{2p^m} each one of $(p^{2n}-1) \div l(p^k-1)$ conjugates; certain systems of $p^{n-m}(p^{2n}-1) \div l(p^k-1)$ conjugate $G_{p^m d_-}$; l systems each of $p^{n-k}(p^{2n}-1) \div l(p^{2k}-1)$ conjugates, of the type of the total binary B_k of determinant 1 in the $GF[p^k]$, k a divisor of n; for n/k even, two systems of groups, each invariant only under itself, of the type $\{B_k, T_\varepsilon\}$, ε a square root of a primitive root of the $GF[p^k]$; for $p^n = 8h \pm 1$, two systems each of $p^n(p^{2n}-1)/48$ conjugate homogeneous tetrahedral G_{24} and two systems each of this number of conjugate homogeneous octahedral G_{48}; for $p^n = 8h \pm 3$, one system of $p^n(p^{2n}-1)/24$ conjugate G_{24}; for $p^n = 10h \pm 1$, two systems each of $p^n(p^{2n}-1)/120$ conjugate homogeneous icosahedral G_{120}; for $p = 5$, n even, two systems each of $5^n(5^{2n}-1)/240$ conjugate G_{120}; for $p = 5$, n odd, one system of $5^n(5^{2n}-1)/120$ conjugate G_{120}.

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[*] Employing *Linear Groups.* p. 284, and foot-note to p. 285, we may show that the groups fall into a single system within $\{\Gamma, T_\varepsilon\}$, where $\varepsilon = \rho^k$. We may show that if $p^n = 5\lambda + 1 = 4t - 1$, there is a single system within the group of binary transformations of determinants ± 1 in the initial field.

For $p = 3$ or 5, G_{24} or G_{120} is also listed under B_k, $k = 1$.

For example, if $p^n = 3$, $\Gamma = G_{24}$ contains only the following subgroups other than identity and itself: an invariant C_2, an invariant di-cyclic G_8, one set of 3 conjugate C_4, one set of 4 conjugate C_3, and one set of 4 conjugate C_6.

Derivation of the Homogeneous from the Fractional Groups.

9. The subgroups G of Γ may be derived from a list of all linear fractional groups of determinant unity.[*] If G is of even order $2g$, it contains T_{-1} and hence may be derived from a fractional group G' of order g by replacing each fractional transformation $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ of G' by the two homogeneous transformations $\begin{pmatrix} \pm a & \pm b \\ \pm c & \pm d \end{pmatrix}$. Next, let G be of odd order. Then G' is of order odd and must (by the list cited) be of one of the following types:

(i) A cyclic group of order $d_{\mp}$, an odd divisor of $\frac{1}{2}(p^n \mp 1)$, generated by $\begin{pmatrix} \delta & 0 \\ 0 & \delta^{-1} \end{pmatrix}$, δ a primitive root of $\delta^d = 1$. If the isomorphic homogeneous $H_{d_{\mp}}$ contained $\begin{pmatrix} -\delta & 0 \\ 0 & -\delta^{-1} \end{pmatrix}$, it would be of order $2d_{\mp}$. Hence H is cyclic and generated by $\begin{pmatrix} \delta & 0 \\ 0 & \delta^{-1} \end{pmatrix}$.

(ii) A commutative group of order p^m composed of certain $\begin{pmatrix} 1 & \mu \\ 0 & 1 \end{pmatrix}$. The homogeneous H_{p^m} can not contain $\begin{pmatrix} -1 & -\mu \\ 0 & -1 \end{pmatrix}$ of period $2p$. Hence H is composed of the $\begin{pmatrix} 1 & \mu \\ 0 & 1 \end{pmatrix}$ with the same range of values for μ.

(iii) A group $G_{p^m d_-}$ given by extending an invariant G_{p^m} by a cyclic C_{d_-}. In view of the preceding cases, H is given by the extension of H_{p^m} by a cyclic H_{d_-}.

From the list cited we now readily obtain the list in § 3. Note that the former list includes a cyclic group of order any divisor $d_{\mp}$ of $\frac{1}{2}(p^n \mp 1)$. If d is

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[*] *Linear Groups*, p. 285. In the long expression for the number of sets of the G_{p^m}, the first factor $p^n - 1$ should be $p^{2n} - 1$. The reference to Professor Moore's original paper should be changed to *Decennial Publications of the University of Chicago*, Vol. IX (1904), pp. 141-190.

odd, we obtain by (i) a homogeneous cyclic group of order d; while by exten-
sion by T_{-1} we obtain a cyclic group of order $2d$, a divisor of $p^n \mp 1$. If d is
odd, we obtain only the latter type. Hence we reach the homogeneous cyclic
$C_{e\mp}$, where e may be any divisor of $p^n \mp 1$.

Invariants of Binary Groups.

10. Let G be a group, of order g, of binary transformations of deter-
minant unity in the $GF[p^n]$. Let $\gamma = 1$ or 2 according as T_{-1} is not or is
contained in G, and set $\omega = g/\gamma$. A point (x, y), in the sense of homogeneous
coordinates, is one of at most ω distinct conjugates under G. A point is called
special if it has fewer than ω conjugates; namely, if it is invariant under some
transformations other than $T_{\pm 1}$ of G. Each system of special points determines
a relative invariant. If we have determined two independent invariants J
and K of degree ω which take on the same factor f_t under each transformation
t of G, then any invariant I which vanishes for no special point is a product
of linear functions of J and K. An integral invariant I with coefficients in
the $GF[p^n]$ is an integral function of J and K with coefficients in that field.[*]

Invariants of the Cyclic and Di-Cyclic Groups.

11. Consider a cyclic group of order d, a divisor > 2 of $p^n - 1$. It is
conjugate within Γ with a C_d generated by T_δ, where δ is a primitive root
of $\delta^d = 1$. The only special points are $(1, 0)$ and $(0, 1)$, the corresponding
invariants being x and y. Now $\omega = d$ or $d/2$ according as d is odd or even.
Further, x^ω and y^ω take on the same factor $\delta^\omega = \delta^{-\omega}$ under T_δ. Hence every
invariant is of the form

$$x^i y^j \prod_k (x^\omega + ky^\omega).$$

Next, consider a cyclic group of order d, a divisor > 2 of[†] $p^n + 1$. It is
conjugate within Γ with a C_d generated by

$$S = \begin{pmatrix} l & -1 \\ 1 & 0 \end{pmatrix}, \quad \rho^2 - l\rho + 1 = 0, \text{ irreducible.} \tag{16}$$

Since S has the canonical from T_ρ, ρ is a primitive root of $\rho^d = 1$. Now S

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[*] *Transactions Amer. Math. Society*, Vol. XII (1911), p. 4.

[†] The present treatment applies also to divisors of $p^n - 1$, but is not as simple as the preceding.

leaves invariant only $(\rho, 1)$, $(\rho^{-1}, 1)$, which are the only special points under G. Hence the invariants are functions of

$$\lambda = x - \rho y, \quad \mu = x - \rho^{-1} y. \tag{17}$$

S multiplies these by ρ^{-1} and ρ, respectively. The invariants with coefficients in the $GF[p^n]$ can be expressed in terms of the absolute invariant $A = \lambda \mu$ and two linear combinations of λ^ω, μ^ω, for example,

$$B = (\lambda^\omega + \mu^\omega)/2, \quad C = (\lambda^\omega - \mu^\omega)/(\rho^{-1} - \rho). \tag{18}$$

Under S these take on the factor $+1$ or -1 according as d is odd or even.

Examples. If $p^n = 3$, $d = 4$, then $l = 0$, $A = x^2 + y^2$, $B = x^2 - y^2$, $C = xy$.

If $p^n = 5$, $d = 3$, then $l = -1$, $A = x^2 + xy + y^2$,

$C = 3(x^2 y + xy^2)$, $B = x^3 - y^3 - 3xy^2 + C/2$.

For $2e$ an even divisor of $p^n - 1$, consider the di-cyclic group

$$G_{4e} = \left\{ T_\epsilon, \quad E = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \right\}, \quad \epsilon^{2e} = 1. \tag{19}$$

The points invariant under a power of T_ϵ are $(0, 1)$, and $(1, 0)$, which are interchanged by E. The corresponding invariant is $Q = xy$. Next $T_a E$ leaves invariant only $(\pm i\alpha^{-1}, 1)$. Now -1 and α are powers of ϵ. The e points

$$P_k^c = (i\epsilon^{c+2k}, 1) \qquad (k = 0, 1, \ldots, e-1) \tag{20}$$

form a system of special points. Indeed, T_ϵ replaces P_k^c by P_{k+1}^c, while E replaces P_k^c by P_{e-c-k}^c. For $c = 0$ or 1, we get the invariants

$$I_c = \prod_{k=0}^{e-1} (x - i\epsilon^{c+2k} y) = x^e - (i\epsilon^c)^e y^e. \tag{21}$$

Under T_ϵ both I_0 and I_1 take on the factor -1; under E, I_0 takes on the factor $-(-i)^e$ while I_1 takes on the factor $+(-i)^e$. We have the identity

$$I_1^2 - I_0^2 = 4i^e Q^e. \tag{22}$$

If* $p^n = 4l - 1$, so that i does not belong to the field, the fundamental system for invariants with coefficients in the field is Q and $I_0 I_1 = x^{2e} - (-1)^e y^{2e}$.

For $2e$ an even divisor of $p^n + 1$, consider the di-cyclic G_{4e} generated by an operator B of period 4 and an operator S of period $2e$, given by (16), where now ρ is a primitive root of $\rho^{2e} = 1$. Thus B is of the form (1) with $\delta = -\alpha$.

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* If $p^n = 4l + 1$, we may replace iy by y and obtain the ordinary dihedron invariants.

Then $SB = BS^{-1}$ if and only if $\gamma = \beta + \alpha l$. Introduce the conjugate imaginary variables (17); then

$$S = \begin{pmatrix} \rho^{-1} & 0 \\ 0 & \rho \end{pmatrix}, \quad B = \begin{pmatrix} 0 & -r \\ r^{-1} & 0 \end{pmatrix}, \quad r = \rho(\beta + \rho\alpha), \quad -r^{-1} = \rho^{-1}(\beta + \rho^{-1}\alpha). \quad (23)$$

In the new variables λ, μ, the only points invariant under the powers of S are $(0, 1)$ and $(1, 0)$, which are interchanged by B. The corresponding invariant is

$$q = \lambda\mu = x^2 - lxy + y^2. \qquad (24)$$

Next, $S^s B$ leaves invariant only $(\pm ir\rho^s, 1)$. Now $-1 = \rho^e$. Hence the invariant point belongs to the system

$$P_k^c = (R\rho^{2k}, 1) \qquad (k = 0, 1, \ldots, e-1),$$

where $R = ir\rho^c$, $c = 0$ or 1. Now S replaces P_k^c by P_{k-1}^c, while B replaces P_k^c by P_{e-c-k}^c. Thus the corresponding invariants are

$$J_c = \lambda^e - R^e\mu^e. \qquad (25)$$

When ρ is replaced by ρ^{-1}, λ and μ are interchanged, while r is replaced by $-r^{-1}$, so that R^e is replaced by its reciprocal. Hence*

$$I_c = (1 - R^e)J_c \qquad (26)$$

remains unaltered and hence belongs to the field (F, i). In case i belongs to F, the fundamental invariants are q, I_0, I_1; in the contrary case, q, I_0I_1. Under S, J_c takes the factor -1; under B the factor $-(i\rho^c)^2$.

For example, let $e = 2$. Taking $l = 0$, we have

$$S = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad B = \begin{pmatrix} \alpha & \beta \\ \beta & -\alpha \end{pmatrix}, \quad \alpha^2 + \beta^2 = -1. \qquad (27)$$

These generate a di-cyclic G_8. We may take† $\alpha\beta \neq 0$. Then

$$q = x^2 + y^2, \quad Q_2 = x^2 - 2\alpha\beta^{-1}xy - y^2, \quad Q_3 = x^2 + 2\beta\alpha^{-1}xy - y^2 \qquad (28)$$

form a fundamental system for G_8. S leaves q unaltered and changes the sign of Q_2 and Q_3, while B leaves Q_2 unaltered and changes the sign of q and Q_3. The relation between the absolute invariants is

$$q^2 + \beta^2 Q_2^2 + \alpha^2 Q_3^2 = 0. \qquad (29)$$

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* We note that R^e is not unity for all the $\nu = p^n + 1$ sets of values of α, β (each set being given by a root of $r^\nu = -1$); in fact, the e values of r which make $R^e = 1$ are $r = -i\rho^{2k-2}$ ($k = 0, \ldots, e-1$). In case e and ν are such that values of r exist for which $R^e = 1$, we may employ the invariants $J_c(\rho - \rho^{-1})$, whose coefficients lie in F.

† If the field contains i, we may take $\beta = 0$ and obtain a simpler system.

Invariants of the Total Group and a Related Group.

12. The group B_n of all binary transformations of determinant unity in the $GF[p^n]$ has (*Transactions, l. c.*) the fundamental system of invariants

$$L = x^{p^n}y - xy^{p^n}, \quad Q = (x^{p^{2n}-1} - y^{p^{2n}-1})/(x^{p^n-1} - y^{p^n-1}). \tag{30}$$

Consider the group $G = \{B_n, T_\varepsilon\}$, were $\varepsilon = \rho^{\frac{1}{2}}$, ρ being a primitive root of the $GF[p^n]$. Then $\varepsilon^{p^n-1} = -1$. Hence T_ε multiplies L and Q by -1. Thus L and Q form a fundamental system for G.

We have noted that, for $p^n = 3$, B_1 is of the homogeneous tetrahedral type G_{24}. The group $G = \{B_1, T_\varepsilon\}$, where $\varepsilon^2 \equiv -1 \pmod{3}$ is of the homogeneous octahedral type. Indeed,

$$S_1 = \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} T_\varepsilon = \begin{pmatrix} \varepsilon & \varepsilon \\ -\varepsilon & \varepsilon \end{pmatrix}, \quad S_2 = \begin{pmatrix} 0 & 1 \\ -1 & -1 \end{pmatrix}, \quad A = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} \tag{31}$$

belong to G and satisfy relations (12) which define G_{48}. Hence, in treating the invariants of G_{24} and G_{48}, we may set $p \neq 3$.

Invariants of the Homogeneous Tetrahedral and Octahedral Groups.

13. In view of the preceding remark, we take $p > 3$. If $p^n = 4l + 1$, so that $\sqrt{-1}$ belongs to the field, we may employ Klein's first[*] form of G_{24}; then the fundamental system of invariants with coefficients in the field is[†] ϕ, ψ, t if $\sqrt{-3}$ belongs to the field, namely, if $p^n = 3k + 1$; but is $\phi\psi$ and t if $p^n = 3k + 2$. The simplicity of this system of invariants rests upon the fact that ϕ and ψ are biquadratics (involving only even powers of the variables). For the outstanding case $p^n = 4l - 1$, in which $\sqrt{-1}$ does not belong to the field F, we proceed to show that no biquadratic (whether involving irrationalities or not) is invariant under a G_{24} with coefficients in F. Indeed, we show that no biquadratic, other than a perfect square,[‡] is invariant under a cyclic transformation S of period 3 with coefficients in F. Let the factors be $x \pm cy$, $x \pm dy$. Then, apart from multiplicative constants, S leaves one factor unaltered and permutes the remaining three. Since S is of period 3, it has the form

$$S = \begin{pmatrix} e & f \\ g & -1-e \end{pmatrix},$$

with the characteristic equation $\omega^2 + \omega + 1 = 0$. Since $p \neq 3$, S leaves no

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* "Ikosaeder," p. 38.
† *Ibid.*, p. 51.
‡ Such a quartic is not invariant under a G_{24}.

linear function absolutely unaltered. Let therefore S multiply $x - cy$ by a cube root ω of unity. The resulting conditions determine e and f. Thus

$$S = \begin{pmatrix} cg + \omega & c\omega^2 - c\omega - c^2 g \\ g & \omega^2 - cg \end{pmatrix}.$$

This replaces $x + cy$ by $x + dy$, where[*]

$$d(2cg + \omega) = 2c\omega^2 - c\omega - 2c^2 g.$$

Since S shall replace $x + dy$ by $x - dy$, we get

$$- d = - c + \omega^2(c + d)/(cg + dg + \omega).$$

Eliminating d, we find that

$$(2cg - \omega^2 + \omega)^2 = - 1.$$

Hence the coefficient $cg + \omega$ in S equals $\frac{1}{2}(- 1 \pm i)$, which is not in F.

Without imposing any restriction on the order p^n of F other than $p > 3$, we determine all sets of generators of G_{24} satisfying relations (11) and such that

$$C = \begin{pmatrix} - 1 & - 1 \\ 1 & 0 \end{pmatrix}. \tag{32}$$

Note that all cyclic C_3 are conjugate under Γ; we choose a simple operator C in order that the quartic invariants shall be simple. Since A shall have period 4,

$$A = \begin{pmatrix} c & d \\ e & - c \end{pmatrix}, \quad B = C^{-1}AC = \begin{pmatrix} d - c & d - 2c - e \\ - d & c - d \end{pmatrix}. \tag{33}$$

The conditions for $C^{-1}BC = AB$ and $|A| = 1$ reduce to

$$e = 1 + d - c, \quad c^2 - cd + d^2 + d + 1 = 0. \tag{34}$$

The points invariant under C are $(\omega, 1)$ and $(\omega^2, 1)$, where ω is a cube root $\neq 1$ of unity. Under G_{24}, $x - \omega y$ and $x + r_i y$ $(i = 1, 2, 3)$ form a conjugate system, where[†]

$$r_1 = \frac{c - \omega^2}{c - d}, \quad r_2 = \frac{c - d}{\omega^2 - d}, \quad r_3 = \frac{\omega^2 - d}{\omega^2 - c}. \tag{35}$$

Evidently $r_1 r_2 r_3 = - 1$. We find that

$$\sigma = r_1 + r_2 + r_3 = (3c^2 d + 2cd + 2c^2 + 2c + d - \omega)/D,$$
$$D = c(c - d)(1 + d) = c + c^2 + c^3,$$

$$\Sigma r_1 r_2 = - \Sigma \frac{1}{r_1} = \sigma - 3.$$

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[*] The coefficient of d and denominator in $- d$ are not zero, since y is not a factor of the quartic by hypothesis.

[†] We may avoid the cases in which a denominator vanishes. Note that $c = 0$ or d requires that $d^2 + d + 1 = 0$ and conversely; that $d = - 1$ requires $c^2 + c + 1 = 0$. We treat later the case in which ω occurs in F.

Hence the invariant quartic is

$$Q = (x - \omega y)[x^3 + \sigma x^2 y + (\sigma - 3)xy^2 - y^3]. \tag{36}$$

If we set $g = \sigma - \omega$, we get

$$Q = x^4 + g x^3 y + (1 - \omega)(g - 2)x^2 y^2 + \omega(4 - g)xy^3 + \omega y^4. \tag{37}$$

If[*] $p^n = 3k + 1$, ω belongs to the field. We may then set $c = 0$ and get

$$A = \begin{pmatrix} 0 & \omega \\ -\omega^2 & 0 \end{pmatrix}, \quad B = \begin{pmatrix} \omega & -1 \\ -\omega & -\omega \end{pmatrix}, \tag{38}$$

$$Q = (x^2 - \omega^2 y^2)[x^2 + \tfrac{4}{3}(1 - \omega^2)xy - \omega^2 y^2]. \tag{39}$$

When Q is known, we can by simple differentiation (Klein, *l. c.*, p. 52) determine another quartic and a sextic invariant.

We may also determine G_{24} so that its invariant G_8 shall be of the simple form generated by (27). The only transformation C of period 3 and determinant 1 for which $SC = CB$, $BC = CSB$, is

$$C = \begin{pmatrix} \tfrac{1}{2}(\alpha - \beta - 1) & \tfrac{1}{2}(\alpha + \beta + 1) \\ \tfrac{1}{2}(\alpha + \beta - 1) & \tfrac{1}{2}(-\alpha + \beta - 1) \end{pmatrix}. \tag{40}$$

Now S leaves invariant only $(\pm i, 1)$, B only $(\alpha \pm i, \beta)$, SB only $(\beta \pm i, -\alpha)$, while, B interchanges the $(\pm i, 1)$ and also the $(\beta \pm i, -\alpha)$, and S interchanges the $(\alpha \pm i, \beta)$ and also the $(\beta \pm i, -\alpha)$. Further, C replaces $(\pm i, 1)$ by $(\alpha \pm i, \beta)$, the latter by $(\beta \mp i, -\alpha)$, and the last by $(\mp i, 1)$. Hence the six points form a system of conjugates under G_{24}. The corresponding invariant is the product of the three quadratic invariants (28) of the G_8. The quartic invariants are now more complicated than (37).

In case $p^n = 8h \pm 1$, 2 is a square, and we may extend G_{24} to G_{48} by either of the transformations $D = \begin{pmatrix} a & -a \\ a & a \end{pmatrix}$, $a^2 = 1/2$, which alone have the properties that D is commutative with S and transforms B into SB.

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[*] In the contrary case we employ suitable products of the invariants derived from (37).

linear function absolutely unaltered. Let therefore S multiply $x - cy$ by a cube root ω of unity. The resulting conditions determine e and f. Thus

$$S = \begin{pmatrix} cg + \omega & c\omega^2 - c\omega - c^2g \\ g & \omega^2 - cg \end{pmatrix}.$$

This replaces $x + cy$ by $x + dy$, where*

$$d(2cg + \omega) = 2c\omega^2 - c\omega - 2c^2g.$$

Since S shall replace $x + dy$ by $x - dy$, we get

$$- d = - c + \omega^2(c + d)/\overline{}cg + dg + \omega).$$

Eliminating d, we find that

$$(2cg - \omega^2 + \omega)^2 = - 1.$$

Hence the coefficient $cg + \omega$ in S equals $\frac{1}{2}(- 1 \pm i)$, which is not in F.

Without imposing any restriction on the order p^n of F other than $p > 3$, we determine all sets of generators of G_{24} satisfying relations (11) and such that

$$C = \begin{pmatrix} -1 & -1 \\ 1 & 0 \end{pmatrix}. \tag{32}$$

Note that all cyclic C_3 are conjugate under Γ; we choose a simple operator C in order that the quartic invariants shall be simple. Since A shall have period 4,

$$A = \begin{pmatrix} c & d \\ e & -c \end{pmatrix}, \quad B = C^{-1}AC = \begin{pmatrix} d - c & d - 2c - e \\ -d & c - d \end{pmatrix}. \tag{33}$$

The conditions for $C^{-1}BC = AB$ and $|A| = 1$ reduce to

$$e = 1 + d - c, \quad c^2 - cd + c^2 + d + 1 = 0. \tag{34}$$

The points invariant under C are $(\omega, 1)$ and $(\omega^2\ 1)$, where ω is a cube root $\neq 1$ of unity. Under G_{24}, $x - \omega y$ and $x + r_i y$ ($i = 1, 2, 3$) form a conjugate system, where†

$$r_1 = \frac{c - \omega^2}{c - d}, \quad r_2 = \frac{c - d}{\omega^2 - d}, \quad r_3 = \frac{\omega^2 - d}{\omega^2 - c}. \tag{35}$$

Evidently $r_1 r_2 r_3 = - 1$. We find that

$$\sigma = r_1 + r_2 + r_3 = (3c^2d + 2cd + \omega^2 + 2c + d - \omega)/D,$$
$$D = c(c - d)(1 + d) = c + c^2 + c^3,$$

$$\Sigma r_1 r_2 = - \Sigma \frac{1}{r_1} = \sigma - 3.$$

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* The coefficient of d and denominator in $-d$ are not zero, since y is not a factor of the quartic by hypothesis.

† We may avoid the cases in which a denominator vanishes. Note that $c = 0$ or d requires that $d^2 + d + 1 = 0$ and conversely; that $d = -1$ requires $c^2 + c + 1 = 0$. We treat later the case in which ω occurs in F.

Hence the invariant quartic is

$$Q = (x - \omega y)[x^3 + \sigma x^2 y + (\sigma - 3)xy^2 - y^3].\tag{36}$$

If we set $g = \sigma - \omega$, we get

$$Q = x^4 + gx^3 y + (1 - \omega)(g - 2)x^2 y^2 + \omega(4 - g)xy^3 + \omega y^4.\tag{37}$$

If[*] $p^n = 3k + 1$, ω belongs to the field. We may then set $c = 0$ and get

$$A = \begin{pmatrix} 0 & \omega \\ -\omega^2 & 0 \end{pmatrix}, \quad B = \begin{pmatrix} \omega & -1 \\ -\omega & -\omega \end{pmatrix},\tag{38}$$

$$Q = (x^2 - \omega^2 y^2)[x^2 + \tfrac{4}{3}(1 - \omega^2)xy - \omega^2 y^2].\tag{39}$$

When Q is known, we can by simple differentiation (Klein, *l. c.*, p. 52) determine another quartic and a sextic invariant.

We may also determine G_{24} so that its invariant G_8 shall be of the simple form generated by (27). The only transformation C of period 3 and determinant 1 for which $SC = CB$, $BC = CSB$, is

$$C = \begin{pmatrix} \tfrac{1}{2}(\alpha - \beta - 1) & \tfrac{1}{2}(\alpha + \beta + 1) \\ \tfrac{1}{2}(\alpha + \beta - 1) & \tfrac{1}{2}(-\alpha + \beta - 1) \end{pmatrix}.\tag{40}$$

Now S leaves invariant only $(\pm i, 1)$, B only $(\alpha \pm i, \beta)$, SB only $(\beta \pm i, -\alpha)$, while B interchanges the $(\pm i, 1)$ and also the $(\beta \pm i, -\alpha)$, and S interchanges the $(\alpha \pm i, \beta)$ and also the $(\beta \pm i, -\alpha)$. Further, C replaces $(\pm i, 1)$ by $(\alpha \pm i, \beta)$, the latter by $(\beta \mp i, -\alpha)$, and the last by $(\mp i, 1)$. Hence the six points form a system of conjugates under G_{24}. The corresponding invariant is the product of the three quadratic invariants (28) of the G_8. The quartic invariants are now more complicated than (37).

In case $p^n = 8h \pm 1$, 2 is a square, and we may extend G_{24} to G_{48} by either of the transformations $D = \begin{pmatrix} a & -a \\ a & a \end{pmatrix}$, $a^2 = 1/2$, which alone have the properties that D is commutative with S and transforms B into SB.

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[*] In the contrary case we employ suitable products of the invariants derived from (37).

The Group of Turns and Slides and the Geometry of Turbines.[*]

By Edward Kasner.

We begin by considering certain simple operations or transformations on the oriented lineal elements of the plane. A *turn* T_a converts each element into one having the same point and a direction making a fixed angle a with the original direction. By a *slide* S_k the line of the element remains the same and the point moves along the line a fixed distance k. These transformations together generate a continuous group of three parameters which we denote by G_3.[†]

Applied to a simply-infinite system of curves, the operation T_a produces a system of isogonal trajectories, while S_k produces equitangential trajectories. Repeating the operations, *i. e.* applying G_3, we obtain from a given system in general ∞^3 new systems. Certain systems, namely those with automorphic transformations, produce fewer trajectorial systems. These are readily determined.

If any transformation G_3 is applied to the ∞^1 elements of a point, the new ∞^1 elements form a configuration which we term a *turbine*. A turbine consists of ∞^1 elements whose points form a circle and whose directions are equally inclined to that circle. The transformations G_3 obviously convert turbines into turbines. There exists a larger group G_{15} of fifteen parameters having this property.

The geometry of turbines is most readily handled by means of a certain representation between the oriented elements of the plane and the points of three-dimensional space, in which turbines correspond to the straight lines of

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[*] Read before the American Mathematical Society, December, 1908.

[†] In any space of constant curvature an analogous group, isomorphic with the displacement group of the space, may be obtained. Thus, for ordinary space the requisite element is the *feuillet*, consisting of a point, line, and plane all incident with one another, and there are three fundamental operations. In a space of variable curvature the operations on the elements generate a group involving in all probability an infinite number of parameters.

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space. Those turbines which are unions, namely, the ∞^3 (oriented) circles, correspond to the lines of a certain linear complex L. In Lie's famous representation of circles by the lines of a complex, the lines not belonging to the complex are not considered; so the concept of turbine does not appear. Furthermore our representation is one-to-one, while Lie's, which employs non-oriented elements, is one-to-two.

We next consider the analogue in the plane of configurations in space which are polar or conjugate with respect to L or the related null-system. In particular we are led to the concept of conjugate differential equations or systems of curves in the plane. This is made use of to prove quite simply and naturally Scheffers' theorems on isogonal and equitangential trajectories.[*] These results are included as special cases of general theorems (numbered 8 and 9 below) concerning systems derived by applying any one-parameter subgroup of G_3, instead of the particular subgroups S and T.

§1.   *The Group G_3.*

All turns T_a constitute a one-parameter group T; and all slides S_k a one-parameter group S. We now prove

THEOREM 1.   *All turns and slides generate a three-parameter group G_3.*

We note in the first place that if any two successions of turns and slides have the same effect upon one lineal element they will have the same effect upon all elements. In the second place we may convert a given element into any other element by a succession of the type

$$T_a S_k T_\beta. \tag{1}$$

Hence a succession of any number of turns and slides will be equivalent to some succession (1). The parameters a, k, β are clearly essential; hence the theorem stated.

For the analytic representation, it will be convenient to define an element, not by the usual (x, y, y'), but by the coordinates (u, v, v') employed by Scheffers: v is the perpendicular from the origin, u is the angle between the perpendicular and the initial line, and the derivative $v' = dv/du$, which we shall also denote by w, is the distance between the foot of the perpendicular and the point of the element.

The slide S_k is then $u_1 = u$, $v_1 = v$, $w_1 = w + k$; we write it simply

$$u, \quad v, \quad w + k. \tag{2}$$

The turn T_a is

$$u + \alpha, \quad v \cos \alpha + w \sin \alpha, \quad - v \sin \alpha + w \cos \alpha. \tag{3}$$

The product (1) is then

$$u + \alpha + \beta, \quad v \cos (\alpha + \beta) + w \sin (\alpha + \beta) + k \sin \beta,$$
$$- v \sin (\alpha + \beta) + w \cos (\alpha + \beta) + k \cos \beta. \tag{4}$$

Introducing new parameters, we may write our group G_3 in the form

$$u + \gamma, \quad v \cos \gamma + w \sin \gamma + a, \quad - v \sin \gamma + w \cos \gamma + b. \tag{5}$$

Since turns and slides are not contact transformations, it is clear that the transformations (5) will not usually be contact transformations. The conditions for a contact transformation give $b = 0$, and $\sin \gamma = 0$; that is, $k \cos \beta = 0$ and $\alpha + \beta = 0$ or π. The resulting transformations are of one of the forms

$$T_{\frac{\pi}{2}} S_k T_{-\frac{\pi}{2}}, \quad T_\pi S_k T_{\frac{\pi}{2}}.$$

The first represents a dilatation D_k; and the second, which may be written $D_k T_\pi$, represents a dilatation accompanied by reversal of orientation.

Hence the only contact transformations in the group G_3 are dilatations, and dilatations with reversal of orientation.

Our group may be written in the simple form

$$S_k D_d T_a. \tag{6}$$

As the independent infinitesimal transformations generating the group, we may take the infinitesimal slide, dilatation, and turn. The symbols of these generators in the Lie notation are respectively

$$\frac{\partial}{\partial w}, \quad \frac{\partial}{\partial v}, \quad \frac{\partial}{\partial u} + w \frac{\partial}{\partial v} - v \frac{\partial}{\partial w}. \tag{7}$$

In terms of the parameters k, d, α in (6), the multiplication of two transformations of the group G_3 is given by the same formulas as arise in the combination of two plane displacements with the usual parameters (x_0, y_0, θ). It is of course easy to see that the two groups are simply isomorphic: each may be regarded as a "group of parameters" in relation to the other group.

Corresponding to the translation subgroup, we have in G_3

$$G_2 \equiv S_k D_d,$$ (8)

which is therefore an invariant subgroup. By these transformations elements are converted into parallel elements, the sense being also preserved. It may be noted that any operation of G_3 is commutative with any displacement.

§ 2. *Systems of Curves with Auto-transformations.*

As a simple application, we may readily find all systems of ∞^1 curves which admit an infinitesimal transformation of the group G_3.

From (7) the general infinitesimal transformation is

$$a \frac{\partial}{\partial u} + (aw + b) \frac{\partial}{\partial v} + (c - av) \frac{\partial}{\partial w}.$$ (9)

If this is to leave invariant the system whose differential equation, in line coordinates, is $w = \dfrac{dv}{du} = f(u, v)$, then

$$a f_u + (af + b) f_v + av - c = 0.$$

The general integration may be avoided by reducing to certain canonical forms.

First assume $a \neq 0$. Then by means of transformations S and D we may reduce (9) to $\dfrac{\partial}{\partial u} + w \dfrac{\partial}{\partial v} - v \dfrac{\partial}{\partial w}$; that is, to a turn. The invariant system must then consist of ∞^1 points. Applying D and S reversed we see that the required system consists of equitangentials of ∞^1 congruent circles; or, what is the same, of isogonals of a set of congruent circles. Next assume $a = 0$. If also $b = 0$, the transformation (9) is simply a slide and the required system consists of ∞^1 straight lines. If $b \neq 0$, it may be reduced to $t = 0$ by means of a turn. Hence the required system consists of isogonals of a set of straight lines.

THEOREM 2. *The only systems of ∞^1 curves which admit an infinitesimal transformation of the group G_3 are those composed of the isogonals, for a fixed angle, of any set of equal circles (including the case of any set of straight lines).*

Given a system of ∞^1 curves, and applying the constructions for isogonal and equitangential trajectories in all possible combinations, we obtain in general ∞^3 new systems. The systems described in the above theorem are distinguished by the fact that they possess fewer than ∞^3 derived systems.*

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*The usual number of derived systems is ∞^2. The number will reduce to ∞^1 if the original system allows two infinitesimal transformations. The only real system of this kind is that composed of parallel straight lines.

Our results may be obtained synthetically by considering the action of the infinitesimal G_3 transformation or the one-parameter group generated by it. Each element is converted successively into the elements of a turbine. In the case of T the paths are points, and in the case of S they are straight lines. *In the general case it is easily seen that the paths are ∞^2 congruent turbines.* The required differential equations are generated by ∞^1 of these turbines.

§3. The General Turbine Group G_{15}.

A turbine is defined as a series of ∞^1 elements obtained from the elements of an oriented circle by means of a turn. The equations of a turbine are of the form

$$v = A \cos u + B \sin u + C, \quad w = B \cos u - A \sin u + D. \qquad (10)$$

The result is a circle if $D = 0$. It is a point if $C = D = 0$. If the base circle of the turbine is a straight line, the analytic representation becomes

$$u = c_0, \quad c_1 v + c_2 w + c_3 = 0.$$

Both turns and slides convert turbines into turbines. We now prove

Theorem 3. *All element transformations converting turbines into turbines constitute a fifteen-parameter group G_{15}.*

The direct attack is rather long but not difficult. The differential equations defining the ∞^4 turbines are

$$\frac{d^2 v}{du^2} = \frac{dw}{du}, \quad \frac{d^2 w}{du^2} = -\frac{dv}{du}. \qquad (11)$$

Expressing the invariance of this set under an infinitesimal transformation

$$\xi \frac{\partial}{\partial u} + \eta \frac{\partial}{\partial v} + \zeta \frac{\partial}{\partial w}, \qquad (12)$$

we find a set of partial differential equations for ξ, η, ζ whose solution gives

$$\xi = [2\alpha v + 2\beta w + a_2 - b_1] \cos u + [2\beta v - 2\alpha w$$
$$+ a_1 + b_2] \sin u + 2\gamma v + 2\delta w + d_1,$$

$$\eta = [\beta(v^2 + w^2) + a_1 v + a_2 w + a_3] \cos u + [-\alpha(v^2 + w^2)$$
$$+ b_1 v + b_2 w + b_3] \sin u + \delta(w^2 - v^2) + 2\gamma vw + c_1 v + c_2 w + d_2, \qquad (13)$$

$$\zeta = [-\alpha(v^2 + w^2) + b_1 v + b_2 w + b_3] \cos u - [\beta(v^2 + w^2)$$
$$+ a_1 v + a_2 w + a_3] \sin u + \gamma(w^2 - v^2) - 2\delta vw - c_2 v + c_1 w + d_3;$$

involving fifteen parameters.

The group G_3 is of course a subgroup of G_{15}. The contact transformations in G_{15}^3 constitute the G_{10} converting circles into circles.

§4. *Representation in Space.*

The group G_{15} may be determined most readily by means of a certain new set of coordinates for oriented elements, namely

$$X = e^{iu}, \quad Y = v - iw, \quad Z = (v + iw)e^{iu}. \tag{14}$$

The equations of the turbine (10) then become linear,

$$Y = \lambda_0 X + \mu, \quad Z = \mu_0 X + \lambda, \tag{15}$$

where

$$\lambda = A + iB, \quad \lambda_0 = A - iB, \quad \mu = C + iD, \quad \mu_0 = C - iD;$$

so that for real turbines λ, λ_0 and μ, μ_0 are pairs of conjugate complex numbers.

We may regard (14) as establishing a correspondence between the elements (u, v, w) of the plane and the points (X, Y, Z) of ordinary space in cartesian coordinates. *In this representation, which we designate by R, the ∞^4 turbines of the plane are pictured by the ∞^4 straight lines of space* (THEOREM 4).

It follows that element transformations converting turbines into turbines must correspond to collineations in space and hence constitute a continuous group G_{15}.

If we define points in space by homogeneous coordinates (X_1, X_2, X_3, X_4), we may write the fundamental representation R in the more symmetric form

$$X_1 : X_2 : X_3 : X_4 = e^{\frac{1}{2}iu} : (v - iw)e^{-\frac{1}{2}iu} : (v + iw)e^{\frac{1}{2}iu} : e^{-\frac{1}{2}iu}. \tag{14'}$$

The element corresponding to a given point of space is

$$u = - i \log X, \quad v = \frac{1}{2}\Big(Y + \frac{Z}{X}\Big), \quad w = \frac{i}{2}\Big(Y - \frac{Z}{X}\Big).$$

Therefore if we consider only the finite elements of the plane and exclude the points for which $X = 0$, the correspondence R is one-to-one.[*]

To an arbitrary curve in space corresponds an arbitrary series of ∞^1 elements in the plane. The series will constitute a union if $dv - w\,du = 0$; that is, if

$$dZ + X\,dY - Y\,dX = 0. \tag{16}$$

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[*] The representation R is closely related and essentially equivalent to the familiar representation of Lie's in which the circles of the plane correspond to the lines of a linear complex. Cf. Lie–Scheffers, "Berührungstransformationen," p. 249. Lie, however, does not orient the elements, so that his correspondence is one-to-two; and turbines are not introduced, so that only lines belonging to the linear complex are represented in the plane.

This Monge equation defines a certain null-system in space. We obtain the same null-system by examining the ∞^3 straight lines of space corresponding to the ∞^3 circles of a plane. The condition that the turbine (14) shall reduce to a circle is $D = 0$; hence the corresponding straight line (15) satisfies the condition $\mu = \mu_0$. This defines a certain linear complex of lines, which we denote by L. The related null-system is of course (16).

THEOREM 5. *Unions in the plane are represented in space by the curves satisfying* (16); *that is, curves whose tangents belong to the fundamental linear complex L. The lines of this complex correspond to the circles of the plane.*

§ 5. *The Operation N. Conjugate Configurations.*

The fundamental linear complex L defines a correspondence such that to each point corresponds a certain plane passing through it (null-system). If any configuration is given in space we may construct the polar or conjugate configuration. Employing our representation R we obtain two configurations in the plane which we also term conjugate. The passage from a plane configuration to its conjugate, we describe as operation N.

The polar of a point in space is a plane passing through it. Hence the conjugate of a given lineal element consists of ∞^2 elements cocircular with it.

Conjugate straight lines in space lead to conjugate turbines. These have the same circle as point locus and the elements are symmetrically related to the elements of the circle. The self-conjugate turbines are of course simply circles.

The conjugate of any series of elements is a new series of elements each of which is cocircular with three consecutive elements of the given series; and vice versa, since the operation N is involutorial.

To a surface in space corresponds by R a field of ∞^2 elements in the plane; that is, a differential equation of the first order with its system of ∞^1 integral curves. If the surface is a plane its image is a parabolic pencil of circles. All the points of an arbitrary surface in the neighborhood of one of its points lie in a plane (tangent plane). *Hence all the elements of an arbitrary field or system of ∞^1 plane curves are cocircular with some definite element*, that is, belong to a definite parabolic pencil of circles; this may be called the approximating or tangent pencil.

We thus arrive at the important concept of conjugate differential equations of the first order, or conjugate systems of curves.

THEOREM 6. *To a general system* Σ *of* ∞^1 *curves in the plane corresponds by the operation* N *a definite conjugate system* $\overline{\Sigma}$. *Those elements of* Σ *in the infinitesimal neighborhood of any one of the elements of* Σ *are cocircular with one of the elements of* $\overline{\Sigma}$; *and vice versa, the relation being mutual.*[*]

The only exception arises in the case where the system Σ has for its image in space a developable surface. The field of elements in this case is generated by ∞^1 turbines each having an element in common with the consecutive turbine. Such a field is peculiar in that the conjugate configuration is not a field but merely a series of ∞^1 elements. A still more special case arises when the image surface is a plane. The field then consists of all the elements cocircular with a fixed element; that is, Σ is a parabolic pencil of circles. In this case the conjugate configuration reduces to the fixed element.

The conjugate of a differential equation, $f(u, v, v') = 0$, is most readily derived by introducing the coordinates X, Y, Z. If the given equation then takes the form $Z = F(X, Y)$, the conjugate equation is obtained in the form $\overline{Z} = \overline{F}(\overline{X}, \overline{Y})$ by eliminating X and Y from

$$\overline{X} = -F_Y, \quad \overline{Y} = F_X, \quad \overline{Z} = F - XF_X - YF_Y, \qquad (17)$$

relations which follow immediately from (19) below.

For any collineation of space there is a definite conjugate collineation with respect to the null-system. Hence for any element transformation G of the group G_{15} of §3 there is a conjugate transformation

$$\overline{G} = N^{-1}GN = NGN \qquad (18)$$

also in the group. If C and $\overline{C}$ are conjugate configurations, so also are CG and $\overline{CG}$. The determination by a construction in the plane follows from the fact that if e and e' are cocircular elements, so also are eG and $e'\overline{G}$. We now prove

THEOREM 7. *The conjugate of a transformation belonging to the group* G_3 *generated by turns and slides, also belongs to that group.*

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[*] If we consider a congruence of lines belonging to L, we see that the two focal surfaces are conjugate with respect to L. Hence in the plane a general congruence of ∞^2 circles may be regarded as composed of the circles of curvature of two distinct systems of ∞^1 curves, Σ and $\overline{\Sigma}$. Cf. Liebmann's discussion in the new edition of Pascal's *Repertorium*, Vol. II, p. 500.

For this purpose we write our polarity in the form of a transformation from the surface element (X, Y, Z, P, Q) to the surface element

$$-Q, P, Z-XP-YQ, Y, -X. \tag{19}$$

In the same coordinates the transformation G_3 is

$$cX, cY+c', Z+c''X, \frac{1}{c}P+\frac{c''}{c}, \frac{1}{c}Q. \tag{20}$$

Calculating $\bar{G}$ from (18), we find that it has the form (20) with new parameters

$$\bar{c}=1/c, \quad \bar{c}'=c''/c, \quad \bar{c}''=c'/c. \tag{21}$$

In terms of the form (6) for the group G_3, it may be shown directly that the *conjugate of the transformation* $S_k D_d T_a$ *is* $S_{-k} D_d T_{-a}$.

In particular the conjugate of T_a is T_{-a}, and of S_k is S_{-k}. Hence if two systems of curves are isogonally or equitangentially related, the same will be true of the two conjugate sets, results equivalent to those of Scheffers. A similar result holds for two systems related by dilatation.

§ 6. *Generalization of Scheffers' Theory of Trajectories.*

Consider a general system Σ of ∞^1 oriented curves. Scheffers applies the one-parameter group of turns on the one hand, obtaining ∞^1 systems (isogonals), and the one-parameter group of slides on the other hand, obtaining ∞^1 new systems (equitangentials). The two sets of results relating to circles of curvature and reciprocity relations for the doubly-infinite systems thus derived are written in parallel columns and a certain (non-projective) duality appears. We may, from our point of view, obtain these results simultaneously as special cases of one theory by starting with *any* one-parameter subgroup G_1 of the group G_3. Applying G_1 to the given system Σ, we obtain ∞^1 new systems or, collectively, a doubly-infinite system Σ_2.[*]

For the group G_1 the path of each element of the plane is a turbine K. There are ∞^2 of these turbines, all congruent: we denote the totality by K_2. To state the results it is necessary to consider also the conjugate group $\bar{G}_1$. The path turbines for this will form a second set of ∞^2 congruent turbines $\bar{K}_2$. The two sets of turbines are built on equal circles, but the angles at which the elements are inclined to the circles, while equal in magnitude, are on opposite sides of the elements of the circles.

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[*] It is assumed of course that the given system Σ is not invariant under G_1.

The general results for any group G_1, which are obtained synthetically or analytically without difficulty, are as follows:

THEOREM 8.   Consider any one of the path turbines K of the set K_2 connected with G_1.   Each element of K determines a curve of the doubly-infinite system Σ_2 generated by applying G_1 to any simply-infinite system Σ.   The locus of the centers of the ∞^1 circles osculating these curves at these elements is a straight line.*   These ∞^1 circles hence touch a certain turbine K' of the set $\bar{K}_2$ conjugate to K_2.†

THEOREM 9.   According to the previous theorem, the system Σ_2 obtained by applying G_1 to Σ induces a definite correspondence between the set of turbines K_2 and the conjugate set $\bar{K}_2$.   There exists another system $\bar{\Sigma}_2$, obtained by applying $\bar{G}_1$ to Σ, for which this correspondence is precisely reversed.

For the group of turns (or slides), the sets K_2 and $\bar{K}_2$ are points (or lines), and we obtain Scheffers' results for isogonals (or equitangentials).   It is to be observed that, in the general case, the sets of ∞^1 osculating circles are linear in the sense of Lie's (higher) geometry of circles, but not usually in the sense of elementary circle geometry.

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* A single exception arises when the group G_1 is the dilatation group D.   The ∞^1 osculating circles are then obviously concentric.

† To a given K will correspond, not its own conjugate $\bar{K}$, but some member of the conjugate set $\bar{K}_2$.   Just which member, is determined by the curves Σ_2.

The Rational Plane Quartic as Derived from the Norm-Curve in Four Dimensions by Projection and Section.

By J. R. Conner.

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§ 1. Introduction.

The theory of the rational plane quartic has been a subject of study among mathematicians for years, but although many interesting properties of the curve have been developed, there is still much work to be done before it can be said to be thoroughly known. Veronese[*] has shown that any rational curve of order n may be regarded as a projection of the rational norm-curve of order n in a space of n dimensions; and similarly that any rational curve of class m may be regarded as a section of a developable of the norm-curve of order m in m dimensions. He calls attention to the usefulness of the method of projection and section as an approach to the study of curves in general, and, in particular, to the study of rational curves. The theory of the rational quartic in space has been treated from this point of view by Marletta,[†] who later applied the same method to the study of the rational quintic in space.[‡] Stahl[§] deduced many properties of the rational plane quartic by projection from space, as an introduction to an extended analytic treatment of the curve in the plane. In this paper I shall deduce some of the properties of the rational quartic in the plane from those of the norm-quartic in four dimensions by projection and section. The method of the paper will be in the main synthetic, though analysis will not be entirely dispensed with. It is not to be hoped that, within narrow limits, a thorough treatment of the subject can be given; I shall be satisfied if in a rather sketchy account I succeed in demonstrating the suggestiveness and fruitfulness of the method.

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[*] Veronese, *Math. Ann.*, 19, p. 208.

[†] Marletta, *Annali di Mat.*, Ser. 3, Vol. VIII, p. 97.

[‡] Marletta, *Rend. Palermo*, Vol. XIX, pp. 94–119.

[§] Stahl, *Journal für Math.*, 101, p. 300.

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Many of the theorems as stated are known. I have not thought it necessary to give minute references for all of these. A great many references to the literature will be found in the works cited. Much use is made throughout of the properties of the rational quartic in space. An historical account of this curve and references are to be found in a work by Berzolari:[*] *Sui Combinanti dei sistemi di forme binarie annessi alle curve gobbe razionale del quart' ordine.*

It may be of interest to the casual reader to know just what of novelty may be expected of the paper. The correlation of invariants and a type of covariants of the plane rational quartic in § 8 is, in the main, evident from the point of view of the plane, but has not, so far as I know, been pointed out in all its generality. The fact that a number of properties of the rational plane quartic may be made to depend on one fundamental idea, as, for instance, the idea of syzygetic lines and planes (§ 9), seems worthy of note. The three-to-one correspondence of § 11 furnishes an example of a class of geometrical transformations that will, I am sure, be used more and more as their properties become better known.[†]

I take this opportunity of expressing my indebtedness to Professors Morley and Coble of the Johns Hopkins University, whose suggestions and encouragement have been invaluable in the preparation of this paper. I wish also to express my appreciation of the financial aid of the Carnegie Institution, without which my presence at this University would have been impossible.

§ 2. *Notation.*

In a space or *flat* of four dimensions, F_4, there exist flats of lower dimensions: F_0, F_1, F_2, F_3. The zero-dimensional flat, F_0, is a point; there are ∞^4 of these in F_4. Similarly, there are respectively ∞^6, ∞^6, ∞^4 F_1's, F_2's and F_3's in F_4. We shall frequently use the words point, line, plane and space instead of the symbols F_0, F_1, F_2 and F_3.

There are two general classes of operations with which we shall deal: *projection* and *section*. We may indicate the operation of projection by the Roman letter, F, with a subscript 0, 1 or 2, the subscript indicating the dimensions of the flat from which the projection is made. Similarly we may indicate the operation of section by the Greek letter Φ, the subscript 1, 2 or 3 referring to the dimensions of the flat by which the section is made. We distinguish flats of the same dimension by superscripts as $F_0^{(1)}$, $\Phi_3^{(a)}$. Thus a projection of a curve C from a point 1 on a plane 2 may be indicated by $C\, F_0^{(1)}\, \Phi_2^{(2)}$.

* Berzolari, *Ann. di Mat.*, Ser. 2, Vol. XX, p. 101.

† Cf. Sturm: *Die Lehre von den geometrischen Verwandtschaften*, Vol. IV, p. 420.

Just as in ordinary space we have systems of ∞^1, ∞^2 and ∞^3 lines, viz., ruled surfaces, congruences, and complexes, so in F_4 we have systems of ∞_4^1, ∞^2, ∞^3, ∞^4 and ∞^5 lines, and the dual systems of ∞^1, ∞^2, ∞^3, ∞^4 and ∞^5 planes. There are two ways in which we may interpret our operations on these: 1) we may ask for the elements of the system incident with a given flat; or 2) we may ask for the result of the operation with the flat on the elements of the system. The first of these we indicate by a dot (.) preceding the sign of operation. Thus, let K be a system of ∞^5 planes in F_4. $K.\Phi_3^{(\alpha)}$ represents the totality of the planes of the system lying in the space α; these touch a surface in α. $K.F_0^{(a)}\Phi_3^{(a)}$ is a complex of lines in α, the traces on α of the planes of the system K which pass through the point a. $K.F_0^{(a)}\Phi_3^{(a)}.\Phi_2^{(\pi)}$ is the complex curve of this complex on the plane π, if π is in α.

We shall use, in the main, the letters $a, b, c, \ldots, x, y, z, \ldots$, as symbols for points, or, with subscripts, as their coordinates; and the letters $\alpha, \beta, \gamma, \ldots$, $\xi, \eta, \zeta, \ldots$, in a similar way for spaces.

We shall consider the norm-quartic in F_4 as fixed throughout our discussion, and shall call it R. Its osculating planes are known to lie on a three-way spread of order 6; we shall call this Σ. Dually we shall use the letter S in referring to the two-way locus of tangent lines of R. This is also of order 6.

If no ambiguity can result, the letters F and Φ indicating projection and section will usually be omitted (§ 4).

§ 3. *Certain Spreads Associated with R in F_4.*

It will be well to consider in the first place a few spreads covariantly connected with R. The norm-curve is the ideal medium for the interpretation of the idea of apolarity, and we shall make extensive use of this notion here. Any F_3 in F_4 meets R in a set of four points, thus defining a quartic in the binary domain on R; dually, through any point of F_4 there are four hyperosculating spaces to R, and a point also defines in this way a binary quartic on R. The apolarity-condition of these two quartics is merely the incidence condition of point and space.

We may give R parametrically by

$$x_0 = 1, \quad x_1 = -t, \quad x_2 = t^2, \quad x_3 = -t^3, \quad x_4 = t^4; \tag{1}$$

or, in spaces,

$$\xi_0 = t^4, \quad \xi_1 = 4t^3, \quad \xi_2 = 6t^2, \quad \xi_3 = 4t, \quad \xi_4 = 1. \tag{2}$$

The quartic determined by any point x may be obtained by substituting in the equation of the point, $(\xi x) = 0$, the parametric values of ξ_i as given by (2). This is

$$x_0 t^4 + 4 x_1 t^3 + 6 x_2 t^2 + 4 x_3 t + x_4 \equiv (x t)^{4 *} = 0, \tag{3}$$

and the quartic determined by any space ξ is

$$\xi_0 - \xi_1 t + \xi_2 t^2 - \xi_3 t^3 + \xi_4 t^4 \equiv (\xi t)^{4 *} = 0. \tag{4}$$

The apolarity-condition of the quartics (3) and (4) is merely the incidence-condition of x and ξ:

$$|\xi x|^4 \equiv \xi_0 x_0 + \xi_1 x_1 + \xi_2 x_2 + \xi_3 x_3 + \xi_4 x_4 = 0. \tag{5}$$

We use the symbol $I_{k, n-k}$ for a binary involution of groups of n points in which k given points in general determine uniquely the remaining $n - k$ of a group.

It follows from (5) that the involution, $I_{3,1}$, of binary quartics apolar to the quartic (3) is cut out of R by spaces on x.

An $I_{1,3}$ of quartics

$$(a t)^4 + \lambda (b t)^4 = 0 \tag{6}$$

is determined by the points of the line joining the points a and b; the involution of quartics apolar to all sets of (6) is

$$(\alpha t)^4 + \lambda (\beta t)^4 + \mu (\gamma t)^4 = 0, \tag{7}$$

where α, β, γ are three linearly independent spaces on the line.

If we put any invariant condition on (3), we have a spread in F_4, the locus of points giving quartics on R for which the given invariant vanishes. The quartic has two invariants, g_2 and g_3. We have the corresponding spreads:

$$g_2 \equiv x_0 x_4 - 4 x_1 x_3 + 3 x_2^2 = 0,$$

$$g_3 \equiv \begin{vmatrix} x_0 & x_1 & x_2 \\ x_1 & x_2 & x_3 \\ x_2 & x_3 & x_4 \end{vmatrix} = 0.$$

g_2 is a quadric spread in S_4, the locus of points defining self-apolar quartics on R.

If the quartics defined by points x and y are apolar, y must be on the space on the four points defined by x, and conversely. We thus have an involutory

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correlation connecting x and y. It is a polarity, *exactly the polarity of the quadric g_2.*[*] For the apolarity-condition of the quartics $(xt)^4$ and $(yt)^4$ is

$$x_0 y_4 + x_4 y_0 - 4 (x_1 y_3 + x_3 y_1) + 6 x_2 y_2 = 0,$$

the polarized form of g_2.

In a polarity in F_4 we have polar point and space, and also polar line and plane, p and π say. π may be regarded as the axis of the pencil of polar spaces of points of p, or as the locus of polar points of spaces on p. Or, polar points and spaces of spaces and points of either p or π are incident with the other. Similarly, polar planes of lines on π contain p, and dually, and polar planes of lines meeting p have a space and a line in common with π (meet π in a line), and dually. Polar lines of planes meeting p meet π, and dually.

Let us next consider g_3: It is the locus of points defining catalectic quartics on R, or the locus of points such that the spaces on them cut out of R *involutions $I_{3,1}$ with a neutral pair.* If yz on R is a neutral pair of the involution defined by a point x, then *any* two points on R may be cut out by a space on xyz; there are therefore ∞^2 spaces on xyz, and x, y and z are on a line. x is therefore on a bisecant line of R, and we have:

g_3 *is the locus of lines bisecant to R.*[†]

Any spread of lines bisecant to a curve in F_4 is only a two-way locus of spaces, the tangent spaces coinciding along a bisecant line. For, let

$$x_i = \phi_i(t) \qquad (i = 0, 1, 2, 3, 4)$$

be a curve. The spread of bisecants is

$$x_i = \phi_i(s) + \lambda \phi_i(t).$$

The tangent F_3 at a point (s, t, λ) is the determinant

$$\left| \; y_i, \quad \frac{d}{ds}\phi_i(s), \quad \frac{d}{dt}\phi_i(t), \quad \lambda\frac{d}{dt}\phi_i(t), \quad \phi_i(s) + \lambda\phi_i(t) \; \right| = 0,$$

which reduces to

$$\left| \; y_i, \quad \phi_i(s), \quad \phi_i(t), \quad \frac{d}{ds}\phi_i(s), \quad \frac{d}{dt}\phi_i(t) \; \right| = 0,$$

and as this is independent of λ our statement is proved. In particular, the hyperosculating space of the curve at a given point contains two consecutive pairs of lines of the spread, and hence must touch it along the tangent line to

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[*] Clifford, *Mathematical Papers*, p. 312.

[†] Cf. Segre, *Mem. Torino*, Ser. 2, Vol. XXXIX, p. 36.

the curve at the point of hyperosculation. g_3 is a two-way of spaces of class 4 as we shall show later.

The spread Σ is the locus of points defining quartics with two equal roots; S is the locus of points defining quartics with three equal roots. Σ is, analytically, the discriminant of $(x\,t)^4 = 0$; S is the complete intersection of g_2 and g_3. Their orders, 6, are obvious from this.

The quadratic invariant of (4) defines a spread of class 2, γ_2. γ_2 is easily seen to be identical with g_2. The cubic invariant of (4) defines a spread, γ_3, of class 3. It is the dual of g_3, and hence must be a two-way of order 4. We reserve its discussion for a later section.

There are ∞^3 planes trisecant to R, and ∞^2 lines in each; lines which carry trisecant planes to R are therefore in a system of ∞^5 lines, a *hypercomplex*. We shall call this the hypercomplex Q. To determine the order of Q, let us consider the surface $Q \cdot F_0^{(a)} \Phi_3^{(a)}$. This is obviously the locus of lines trisecant to $R\,F_0^{(a)} \Phi_3^{(a)}$. The latter curve is a rational quartic in a, and its trisecant lines are known to lie on a unique quadric. It follows that the spread of lines of Q on a, $Q \cdot F_0^{(a)}$, is of order 2, and we have

Lines carrying planes trisecant to R are in a quadratic hypercomplex.

The tangent line to R at any point t may be written

$$y_i = x_i + \lambda \frac{dx_i}{dt};$$

or, homogeneously,

$$y_i = \tau_1 \frac{\partial x_i}{\partial t_1} + \tau_2 \frac{\partial x_i}{\partial t_2}, \tag{8}$$

where x_i have the values given in (1). The line (8) meets any space ξ in the point determined by the equation

$$(\xi y) = \tau_1 \frac{\partial (\xi t)^4}{\partial t_1} + \tau_2 \frac{\partial (\xi t)^4}{\partial t_1} = 0,$$

whence we may take

$$\tau_1 = \frac{\partial (\xi t)^4}{\partial t_2}, \quad \tau_2 = -\frac{\partial (\xi t)^4}{\partial t_1},$$

and

$$y_i = \frac{\partial x_i}{\partial t_1} \frac{\partial (\xi t)^4}{\partial t_2} - \frac{\partial x_i}{\partial t_2} \frac{\partial (\xi t)^4}{\partial t_1} = J\,[x_i,\ (\xi t)^4], \tag{9}$$

where we indicate by J the Jacobian of x_i and $(\xi t)^4$.

The equations (9) *are a parametric representation of the rational sextic curve cut out of S by the space ξ; using our operational symbols, it is $S\Phi_3^{(\xi)}$.*

If $(\xi t)^4$ is the point t_1 taken four times, the equations (9) have the common factor $|t t_1|^3$, and (9) thus reduces to a rational cubic. ξ is in this case a hyper-osculating space of R, or, as we may say, a space of R. (9) is now only the polarized forms of the values of x as to t_1. We have therefore the theorem:

The first osculants of R are cut out of S by the spaces of R.[*]

The parametric representation of the curve $\Sigma \Phi_3^{(r)}$ is also easily found. The osculating plane to R at the point t is

$$y_i = \lambda \frac{\partial^2 x_i}{\partial t_1^2} + \mu \frac{\partial^2 x_i}{\partial t_1 \partial t_2} + \nu \frac{\partial^2 x_i}{\partial t_2^2},$$

where λ, μ, ν are homogeneous variable parameters. This meets the plane π which, let us say, is defined by the pencil of spaces

$$(\xi x) + \lambda (\eta x) = 0, \tag{10}$$

in the points given by

$$y_i = \begin{vmatrix} \dfrac{\partial^2 x_i}{\partial t_1^2}, & \dfrac{\partial^2 x_i}{\partial t_1 \partial t_2}, & \dfrac{\partial^2 x_i}{\partial t_2^2} \\[2ex] \dfrac{\partial^2 \xi}{\partial t_1^2}, & \dfrac{\partial^2 \xi}{\partial t_1 \partial t_2}, & \dfrac{\partial^2 \xi}{\partial t_2^2} \\[2ex] \dfrac{\partial^2 \eta}{\partial t_1^2}, & \dfrac{\partial^2 \eta}{\partial t_1 \partial t_2}, & \dfrac{\partial^2 \eta}{\partial t_2^2} \end{vmatrix}, \tag{11}$$

where we have written ξ for $(\xi t)^4$ and η for $(\eta t)^4$. A curve (11) is evidently completely defined as soon as we give the pencil of spaces (10) or the pencil of binary quartics on R which it defines.

§ 4. *Section by a Space.*

In the operations of projection with which we shall deal only points and lines are involved as the elements from which we project; similarly in the operations of section we are concerned with planes and spaces as the elements by which the operation is determined. No ambiguity can result if we omit the symbols of projection and section, F and Φ, and write merely the symbol of the element involved in the operation; thus $R p \pi$ is the result of projecting R from a line p on a plane π. The dot (.) will be used in the sense previously explained to indicate that we are asking for the elements of a given system of planes or lines incident with the element by which the operation is determined.

The curve $S\alpha$ is the dual of $R a \alpha$. $S\alpha$ is a curve in the space α of order 6 and class 4; it is a rational quartic of planes. Its four cusps are obviously the

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[*] Cf. Berzolari, *Annali di Mat.*, Ser. 2, Vol. XXI, p. 7.

four points $R\alpha$. It is the complete intersection of the cubic surface $g_3\alpha$ and the quadric surface $g_2\alpha$. Let b_t, p_t, π_t, β_t be the point, plane, line, and space, respectively, of R at the point t. Then $p_t\alpha$ is a point of $S\alpha$; we may name this also with the parameter t. $\pi_t\alpha$ and $\beta_t\alpha$ are the line and plane of $S\alpha$ at the point t. $g_2\alpha$ and $g_3\alpha$ are the loci of points whose four planes of $S\alpha$ give quartics respectively self-apolar and catalectic. Since β_t touches g_3 along the line p_t, it follows that $\beta_t\alpha$, the plane of $S\alpha$ at the point t, touches $g_3\alpha$ at this point; *i. e.*, $S\alpha$ is an asymptotic curve on $g_3\alpha$.[*]

We have seen that the cubic osculants of R are the curves $S\beta_t$. If a rational curve is projected into a lower space, its osculants are projected into the osculants of the new curve. It follows that the osculants of $R\alpha\alpha$ are the curves $S\beta_t\alpha\alpha$.

Dually, the cubic osculant t of $S\alpha$ is the envelope of the system of planes $\pi_{t'}b_t\alpha$; in other words, the envelope of the system of planes into which the planes $\pi_{t'}$ of R are projected from a point t of R. It follows:

The osculants of $S\alpha$ considered as point-loci are the curves $Rb_t\alpha$.

Rb_t is a cubic cone projecting R from the point b_t of it. It follows that Rb_t is on g_3. Or:

$g_3\alpha$ is the locus of the curves $Rb_t\alpha$, the cubic osculant of $S\alpha$ considered as point-loci.[†]

It is easily seen that $Rb_t\alpha$ is on the cusps of $S\alpha$, touches $S\alpha$ at the point t and has the same osculating plane with $S\alpha$ at t. Two osculants $Rb_t\alpha$ and $Rb_{t'}\alpha$ meet in one and only one point of α apart from the four points $R\alpha$, namely, the point $b_tb_{t'}\alpha$. Three osculants of $S\alpha$, $Rb_t\alpha$, $Rb_{t'}\alpha$, $Rb_{t''}\alpha$, meet, two by two, in three points of a line, the line $b_tb_{t'}b_{t''}\alpha$. Now $b_tb_{t'}b_{t''}$ is a plane trisecant to R, and $b_tb_{t'}b_{t''}\alpha$ is a line on this plane. This gives us an interpretation of the complex $Q.\alpha$ in α:

The three points of intersection, two by two, of three osculants of $S\alpha$ are on a line. Such lines are in the quadratic complex $Q.\alpha$.

The planes of the tetrahedron $R\alpha$ are singular planes of $Q.\alpha$, since obviously every line on one of these planes carries a trisecant plane to R. Hence:

$Q.\alpha$ is a tetrahedral complex; the points $R\alpha$ are the vertices of the fundamental tetrahedron.

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* Segre, *loc. cit.*   † Study, *Leipz. Berichts*, 1886, p. 9.

If π is a plane in α the conic $Q . \pi$ touches the lines cut out of the faces of this tetrahedron by π.

$g_3 \alpha$ contains the six lines joining the four vertices of this tetrahedron, and is therefore a four-nodal cubic surface, with the points $R\alpha$ for nodes.[*] The conic $Q . \pi$ therefore touches four lines whose vertices are on the cubic curve $g_3 \pi$. There must be an infinity of such four-lines, one defined by every space α on π.

The four tangent planes from a line of α to $g_3 \alpha$ are on the tangent spaces from this line to g_3. It follows:

g_3, *in spaces, is a two-way of class* 4.

Tangent spaces to g_3 may be regarded as spaces bitangent to R. Dually we have:

The double spread of Σ is a two-way, γ_3, of order 4, and a three-way of class 3.

γ_3 may be regarded as the locus of points of intersection of pairs of planes of R.

§5. *Mapping of $g_3 \alpha$ on a Plane.*

Some light is shed on the figures with which we are dealing by the mapping of $g_3 \alpha$ on a plane, π. We can represent plane sections of $g_3 \alpha$ by cubics of the threefold linear system Ψ on the six points of intersection of four lines 1, 2, 3, 4 in π. The six points may be named 12, 13, 14, 23, 24, 34. Now any cubic of Ψ meets any of the four lines, 1 say, in three points 12, 13, 14, and hence, if a cubic Ψ is forced on another point of this line, it must degenerate into 1 and a conic on 23, 34, 42. In other words, a plane section of $g_3 \alpha$ on the map of *any* point of 1 is on the map of *every* point of 1, and it follows that 1 is represented by a single point of $g_3 \alpha$. That this point is a node of $g_3 \alpha$ follows from the fact that conics on 34, 42, 23 meet 1 twice, and hence every plane section on the point 1 of $g_3 \alpha$ passes twice through that point. 1, 2, 3, 4, therefore, map into nodes of $g_3 \alpha$. The points 12, etc., give the six lines joining the nodes. The lines of the diagonal triangle of 1, 2, 3, 4 map into three lines on a plane, the three further lines of $g_3 \alpha$. These lines are important and use will be made of them later.

Consider three points x, y, z of π which, with the six points 12, etc., are base-points of a pencil of cubics. Such a set of three points maps into the three

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[*] Segre, *loc. cit.*

points of intersection of $g_3 \alpha$ with a line p in α. Now x, y, z must be on a conic, with 23, 34, 42, the map of this conic on $g_3 \alpha$ being the plane section joining the node 1 to the line p. It follows that 2, 3, 4, xy, yz, zx are six lines of a conic, C. By a similar argument 1, 2, 3, xy, yz, zx are six lines of a conic, necessarily the same conic, C. xy, yz, zx therefore touch a conic of the range on 1, 2, 3, 4. Therefore we have

The three meets of $g_3 \alpha$ with a line are maps of the vertices of a triangle circumscribed to a conic of the range on 1, 2, 3, 4.

It follows easily from this that the maps of the conics of this range are the asymptotic curves of $g_3 \alpha$. $S \alpha$ is therefore the map of a conic, C, of the range on 1, 2, 3, 4. The points of contact of C with 1, 2, 3, 4 give the cusps of $S \alpha$.

A first osculant of $S \alpha$ is a space cubic on the cusps $R \alpha$ and touching $S \alpha$ at a definite point, being thereby uniquely determined. Hence:

First osculants of $S \alpha$ are the maps of tangents to C. The tetrahedral complex $Q . \alpha$ is represented on π by triangles circumscribed to C, the vertices of such a triangle mapping into the three points in which a line of $Q . \alpha$ meets $g_3 \alpha$.

This last theorem is the object of this section; the representation of the four-nodal cubic surface on a plane is well-known.

§ 6.   *The Curve $\Sigma \pi$.*

$\Sigma \pi$ is the dual of $R p \pi$, and is a rational curve of order 6 and class 4. If β_t is the space t of R, we call the line $\beta_t \pi$ the line t of $\Sigma \pi$. The point t of $\Sigma \pi$ is $\pi_t \pi$, π_t being the plane of R at the point t. $S \pi$ is a group of six points, the intersections of $g_2 \pi$ and $g_3 \pi$. From a point of the group $S \pi$ three spaces to R and hence three tangents to $\Sigma \pi$ coincide. The points $S \pi$ are therefore cusps of $\Sigma \pi$.

Spaces on π meet R in sets of a pencil $(I_{1,3})$ of quartics

$$(\alpha t)^4 + \lambda (\beta t)^4 = 0, \tag{1}$$

where α and β are two independent spaces on π. The space $\alpha + \lambda \beta = 0$ is incident with every point of π. It follows that all quartics of the pencil (1) are apolar to all quartics determined on R, or $\Sigma \pi$, by points of π, and therefore (1) is the fundamental involution of $\Sigma \pi$. Hence:

Quartics of the fundamental involution of $\Sigma \pi$ are cut out of R by spaces on π.

It will be of advantage in the study of the curve $\Sigma \pi$ to consider the spreads determined by R on a space α on π. It should be remembered, however, that

in this way we isolate a set of the fundamental involution on $\Sigma\pi$, and hence all curves thus obtainable on π are not intrinsic to $\Sigma\pi$ alone. We have considered in a previous section some of the geometrical forms determined by R on a space α.

$-\Sigma\alpha$, the developable of $S\alpha$, meets π in $\Sigma\pi$. The cusps of $\Sigma\pi$ are the six points where $S\alpha$ meets π. The osculants of $\Sigma\pi$ are the intersections with π of the developables of the osculants of $S\alpha$. We see at once that

$\qquad$ *The curve $g_3\pi$ may be regarded as the locus of cusps of cubic osculants of $\Sigma\pi$.*

There are a pair of lines of $Q.\alpha$ through any point, x, of $g_3\pi$ and on π; they touch the conic $Q.\pi$. Now we have shown that the lines $Q.\alpha$ are lines bisecant to cubic osculants of $S\alpha$; these two lines therefore pass through the two further cusps of the osculant to $\Sigma\pi$ which has a cusp at x. It follows that

$\qquad$ *The conic $Q.\pi$ is the locus of cusp three-lines of cubic osculants of $\Sigma\pi$, or $Q.\pi$ is the Stahl conic K of $\Sigma\pi$.*[*]

The determination given above for cubic osculants of $\Sigma\pi$ from the point-forms of the osculants of $S\alpha$ does not apply if a_t is a point of the set $R\alpha$. But, regarding the cubic osculants of $S\alpha$ as the systems of planes $\pi_{t'}a_t\alpha$ for varying t', $\pi_{t'}$ and a_t being plane and point of R at t' and t respectively, it is obvious that the osculant at a point t of the set $R\alpha$ is the quartic cone with three cuspidal generators which projects $S\alpha$ from the point t. We thus obtain proper osculants of $\Sigma\pi$ for the points corresponding to $R\alpha$, as we should. The grouping of the cusps of the four osculants to $\Sigma\pi$ at four points of a set of the fundamental involution and the one-to-one relation of the conic K to $\Sigma\pi$, are geometrically obvious here.

The hypercomplex dual to Q is a quadratic system of ∞^5 planes, Q' say. There is a double infinity of planes of Q' on any space α. They touch a quadric which we may call $Q'.\alpha$. This quadric is the locus of lines carrying three spaces of R or three planes of $S\alpha$. $Q'.\alpha$ is therefore the unique quadric touched by planes of $S\alpha$. If

$$(\alpha t)^4 = 0$$

is a quartic giving the four points $R\alpha$, then $(\alpha t)^4$ is a set of the fundamental involution of $\Sigma\pi$; the sets of three planes of $S\alpha$ on lines of $Q'.\alpha$ define, as is known, on $S\alpha$ the apolar cubics of $(\alpha t)^4 = 0$. On any point of the conic $Q.\alpha\pi$ there are four planes of $S\alpha$ and their traces on π give the four lines of $\Sigma\pi$ on this point. Three of these planes meet in a line of $Q'.\alpha$. By means of the

* Stahl, *Journal für Math.*, 101, p. 304.

fourth plane $Q' . \alpha \pi$ is in one-one correspondence with $\Sigma \pi$ and the point of $Q' . \alpha \pi$ is incident with its corresponding line of $\Sigma \pi$. $Q' . \alpha \pi$ is therefore a *perspective conic* of $\Sigma \pi$, that particular perspective conic associated with the definite set $(\alpha t)^4 = 0$ of the fundamental involution of $\Sigma \pi$.* It may be defined as the locus of points in π three of whose tangents to $\Sigma \pi$ define an apolar cubic of $(\alpha t)^4$.

We have said that the two-way γ_3, the double spread of Σ, is of order 4 and class 3. It follows that the surface $\gamma_3 a \alpha$ is a surface of order 4 and class 3, a Steiner quartic surface. Every space section of γ_3 must be a rational quartic since every plane section of a Steiner surface is a rational quartic. The curve $\gamma_3 \alpha$ is the double curve of $\Sigma \alpha$, the developable of $S\alpha$. $\gamma_3 a \pi$, or merely $\gamma_3 \pi$, are the four double points of $\Sigma \pi$.

The quartic curve $\gamma_3 \alpha$ lies on a unique quadric surface in α. This quadric surface meets π in a conic on the four nodes of $\Sigma \pi$, and we shall see later that it cuts out of $\Sigma \pi$ in addition the set of the fundamental involution associated with the space α on π. It may be regarded as the locus of lines in α trisecant to γ_3. We see thus that planes on lines trisecant to γ_3 are in a quadratic system. This system and its dual hypercomplex of lines are important and we shall denote a later section to their consideration.

§7. *Projection of R from a Line.*

The projection of R from a line p on a plane π, in our notation the curve $Rp\pi$, is a projectively general rational quartic on the plane π. Using a former notation for points and lines of R, the point $a_t p \pi$ is the point t of $Rp\pi$ and the line $p_t p \pi$ is the tangent to $Rp\pi$ at this point.

Spaces on p meet π in lines. They cut out of R the sets of parameters of intersections of lines with $Rp\pi$. The fundamental involution of $Rp\pi$ is defined by the scheme of points common to all of these spaces; that is, by the points of p. For convenience we shall consider π as the polar plane of p as to g_2 or as to R. We have then that the fundamental involution of $Rp\pi$ is defined either by points of p or by spaces on π. Dually the fundamental involution of $\Sigma \pi$ is defined by the same scheme of points or spaces. The

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*Two plane curves, one given in lines and the other in points, are said to be *perspective* if they are in one-one correspondence and corresponding point and line are incident. See Stahl, *Math. Ann.*, 38, pp. 561 and 575. A rational plane quartic has ∞^3 perspective cubics, one being determined uniquely by an arbitrary binary cubic; it has ∞^1 perspective conics, one associated with each set of the fundamental involution. Compare Coble, AMERICAN JOURNAL OF MATHEMATICS, Vol. XXXII, p. 352.

involution defined on $\Sigma \pi$ by tangents through a point of π is determined on R either by points of π or by spaces on p.

The polar point of any space on p as to g_3 is on π. The polar point of the space joining p to any line p' of π is the polar point of p' as to $g_2 \pi$. If p' touches $R p \pi$, two roots of the quartic cut out of R by the space pp' coincide. The polar point of the space pp' must therefore be on Σ, since Σ is the locus of points of F_4 defining quartics on R with a double root. It follows:

The curves $R p \pi$ and $\Sigma \pi$ are polar reciprocals as to the conic $g_2 \pi$.

$g_2 \pi$ is the locus of points defining self-apolar quartics on $\Sigma \pi$ and the locus of lines cutting out self-apolar quartics from $R p \pi$. It touches the stationary lines of $R p \pi$ and is on the stationary points of $\Sigma \pi$.

An interesting system of configurations is determined on π by projecting from p the six lines joining the four points $R a$ where a is a space on π. The four planes on the points $R a$ meet π in lines of the Stahl conic K of $\Sigma \pi$. The six lines on $R a$ meet π in the intersections of these four tangents to this conic; these are points of $g_3 \pi$. Projecting from p, the points $R a p \pi$ are a set of the fundamental involution on $R p \pi$. The projection gives us the following theorem:

The six lines joining the four points of a set of the fundamental involution on $R p \pi$ meet $g_3 \pi$ in six points of a four-line. These four lines touch the Stahl conic K of $\Sigma \pi$, determining on the latter conic, which is in natural one-one correspondence with $R p \pi$, the set of the fundamental involution.

We may point out that the curves K and $g_3 \pi$ are the polar reciprocals of the dual curves as determined by $R p \pi$ as to $g_2 \pi$.

The locus of lines in F_4 joining points in which spaces on a given plane π meet R is a ruled two-way of order 9. For any one of the varying spaces meets this two-way in the curve $g_3 \pi$ and the six lines. This nonic two-way is elliptic, since its lines are in one-one correspondence with $g_3 \pi$. Its lines project from p on π into the lines of a curve of class 9. Hence:

The lines of complete four-points of sets of the fundamental involution on $R p \pi$ touch a curve of class 9 and genus 1.

We may also state the more general theorem:

Any involution

$$(\gamma\, t)^4 + \lambda (\delta\, t)^4 = 0$$

on $R p \pi'$ determines in the same way a cubic and a conic bearing a relation similar

to the above to the *four-points* of the involution. *Lines of complete four-points of the involution on* $R\,p\,\pi$ *touch a curve of class* 9 *and genus* 1.

For this involution determines a definite plane π'. Using this as the plane in $R\,p\,\pi'$ we have the theorem as we have stated it.

§ 8.   *Connection of Invariants of* $R\,p\,\pi$ *and a Certain Type of Covariants of* $\Sigma\,\pi$.

Let α, β, γ be three independent spaces on the line p from which we are projecting. We may at once take as coordinates in the plane π.

$$\left.\begin{aligned}
X_0 &= (\alpha\,x), \\
X_1 &= (\beta\,x), \\
X_2 &= (\gamma\,x).
\end{aligned}\right\} \tag{1}$$

The quartic $R\,p\,\pi$ is then

$$\left.\begin{aligned}
X_0 &= (\alpha\,t)^4, \\
X_1 &= (\beta\,t)^4, \\
X_2 &= (\gamma\,t)^4.{}^*
\end{aligned}\right\} \tag{2}$$

The projection of any point of F_4 from p on π is given at once by substitution of its coordinates in (1). We define the coordinates of the line in π joining any two points X, Y as the determinants of the matrix

$$\left\|\begin{array}{ccc} X_0, & X_1, & X_2 \\ Y_0, & Y_1, & Y_2 \end{array}\right\|, \tag{3}$$

whence the projection on π of the line

$$p_{ij} \equiv x_i\,y_j - x_j\,y_i$$

is

$$\left.\begin{aligned}
\Xi_0 &= \Sigma \mid \beta\gamma \mid_{ij} p_{ij}, \\
\Xi_1 &= \Sigma \mid \gamma\alpha \mid_{ij} p_{ij}, \\
\Xi_2 &= \Sigma \mid \alpha\beta \mid_{ij} p_{ij}
\end{aligned}\right\} \tag{4}$$

from (1) and (3), where we have written

$$\mid \beta\gamma \mid_{ij} = \beta_i\gamma_j - \beta_j\gamma_i,$$

and Σ is used as a summation-sign. If the line in F_4 is given by three spaces ξ, η, ζ on it, we have

$$\rho\,p_{ij} = \pi_{klm} = \left|\begin{array}{ccc} \xi_k & \xi_l & \xi_m \\ \eta_k & \eta_l & \eta_m \\ \zeta_k & \zeta_l & \zeta_m \end{array}\right|,$$

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* Cf. Veronese, *Math. Ann.*, 19, *loc. cit.*

and (4) becomes

$$\Xi_0 = \begin{vmatrix} \beta_0, & \beta_1, & \beta_2, & \beta_3, & \beta_4 \\ \gamma_0, & \gamma_1, & \gamma_2, & \gamma_3, & \gamma_4 \\ \xi_0, & \xi_1, & \xi_2, & \xi_3, & \xi_4 \\ \eta_0, & \eta_1, & \eta_2, & \eta_3, & \eta_4 \\ \zeta_0, & \zeta_1, & \zeta_2, & \zeta_3, & \zeta_4 \end{vmatrix}, \tag{5}$$

with similar expressions for Ξ_1 and Ξ_2.

If we require an invariant of $Rp\pi$ to vanish, we are obviously imposing a single condition on the line p from which we are projecting R. Now, if a single condition is imposed on a line in F_4, the line is in a hypercomplex. We have then,

Invariants of $Rp\pi$ are represented in F_4 by hypercomplexes of lines associated with R.

Any hypercomplex in F_4 is a function of determinants of the type

$$| \; \xi_k, \; \eta_l, \; \zeta_m \; |,$$

where ξ, η, ζ are spaces on a line. In the plane π this of course is equivalent to the theorem that any invariant of a rational curve is a function of determinants of the matrix of coefficients of the binary forms giving the parametric representation of the curve. Any invariant of $Rp\pi$, in our notation, is a function of determinants of the type

$$| \; \alpha_j, \; \beta_k, \; \gamma_l \; |,$$

such a function defining a hypercomplex in F_4. Now by means of the polarity associated with R, or, which is the same thing, by means of the polarity of the quadric g_2, we have associated with every hypercomplex of lines in F_4 a hypercomplex of planes; that is, a system of planes satisfying one condition. Now every line in F_4 carries ∞^2 planes; if we ask for the planes on a line that lie in a hypercomplex of planes, we have only ∞^1 planes; and the locus of these planes is a cone-spread of the same order as the hypercomplex, and having the line as vertex. Every hypercomplex of planes in F_4 covariantly associated with R obviously determines a covariant curve of $Rp\pi$ when we project from p, the planes of the hypercomplex on p tracing out on π the covariant curve of $Rp\pi$ in question.

We now show how, given a hypercomplex of planes in F_4, we may obtain the equation of the corresponding covariant of $Rp\pi$. Let

$$F(\; | \; z_k, \; y_l, \; x_m \; | \;) = 0$$

be the equation of any covariant hypercomplex in F_4. If a and b are any two points on the line p, the equation of the hypercomplex cone determined by p is

$$F(\mid a_k, \ b_l, \ x_m \mid) = 0.$$

Let α, β, γ be three independent spaces on p. Then

$$(\alpha a) = 0, \quad (\alpha b) = 0, \quad \rho X_0 = (\alpha x),$$
$$(\beta a) = 0, \quad (\beta b) = 0, \quad \rho X_1 = (\beta x),$$
$$(\gamma a) = 0, \quad (\gamma b) = 0, \quad \rho X_2 = (\gamma x).$$

These nine equations are all possible bilinear relations among the rows of the two matrices

$$\left\| \begin{matrix} \alpha_0, & \alpha_1, & \alpha_2, & \alpha_3, & \alpha_4, & X_0 \\ \beta_0, & \beta_1, & \beta_2, & \beta_3, & \beta_4, & X_1 \\ \gamma_0, & \gamma_1, & \gamma_2, & \gamma_3, & \gamma_4, & X_2 \end{matrix} \right\|$$

and

$$\left\| \begin{matrix} a_0, & a_1, & a_2, & a_3, & a_4, & 0 \\ b_0, & b_1, & b_2, & b_3, & b_4, & 0 \\ x_0, & x_1 & x_2, & x_3, & x_4, & -\rho \end{matrix} \right\|,$$

whence follows the proportionality of the corresponding determinants

$$\sigma \mid a_k, \ b_l, \ x_m \mid \ = \ \mid a_i, \ a_j, \ X \mid, \tag{6}$$

where σ is a proportionality factor, and we have

$$F(\mid a_k, \ b_l, \ x_m \mid) = M F(\mid a_i, \ a_j, \ X \mid), \tag{7}$$

where M is merely a power of the σ in (6) and does not vanish.

The left-hand member of (7) is an invariant of the section of Σ by the plane $a\,b\,x$; i. e., it is an invariant of the quartic

$$\Xi_0 = (a\,t)^4,$$
$$\Xi_1 = (b\,t)^4,$$
$$\Xi_2 = (x\,t)^4,$$

a quartic given with binomial coefficients. Hence, we have the following:

If, for a rational quartic given without binomial coefficients, we have a covariant curve whose equation is expressible in terms of determinants of the type

$$\mid a_i \ a_j \ X \mid,$$

and we replace each such determinant by the complementary determinant

$$\mid a_k \ b_l \ c_m \mid,$$

where the a, b, c are coefficients of binary quartics written with binomial coefficients

which give the parametric representation of a second rational quartic, we obtain an invariant of the latter quartic.

We have thus a one-one correspondence between covariants of this special type and invariants of $Rp\pi$. It is easy to discover, geometrically, the corresponding covariant of some of the simpler invariants.

We give in the following table some invariants and corresponding covariants, where both quartics are understood as quartic point-loci. Some of the covariants are more easily defined as line-loci, but they are of course obtained as point-loci from invariants by the above translation principle.

Invariants.	*Covariants.*
I. Triple point condition.	Stahl conic K.
II. Undulation condition.	Quartic (in points).
III. Cusp condition.	Product of stationary lines.
IV. Tac-node condition.	Locus of lines cutting out harmonic pairs from the quartic.
V. Condition that two self-apolar sets of fundamental involution coincide.	Locus of lines cutting out self-apolar quartics from the quartic curve.
VI. Condition of self-apolarity of sextic giving the points of inflection.	Locus of points from which six tangents define a self-apolar sextic.
VII. Condition that two points of inflection belong to a set of the fundamental involution (discriminant of the conic on the points of inflection).	Product of the four double-tangents.
VIII. Flecnode condition.	Locus of points of inflection of cubic osculants.
IX. Condition that three double tangents be on a point.	Conic on the six points of inflection.

This list might be extended indefinitely. It will be obvious from the proofs of these facts that a geometrical characterization of an invariant is, in a sense, a definition of the corresponding covariant.

For convenience the proofs are given for the dual quartics $Rp\pi$ and $\Sigma\pi$. An invariant of $Rp\pi$ defines a hypercomplex C. The curve $C.\pi$ is the corresponding covariant curve of $\Sigma\pi$.

I. The three nodes of $Rp\pi$ are determined by the three lines of g_3 which meet p. If $Rp\pi$ have a triple point, p must carry a plane trisecant to R. The hypercomplex of lines thus defined gives on π the Stahl conic K of $\Sigma\pi$.

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II. If $R p \pi$ have an undulation, p is on a space of R. These spaces, as we have seen, mark out on π the lines of $\Sigma \pi$.

III. If $R p \pi$ have a cusp, p meets S. Lines in π meeting S are lines through one of the six cusps of $\Sigma \pi$.

IV. If $R p \pi$ have a tac-node, two nodes have come together and p touches g_3. The locus of lines in π which touch g_3 is the curve $g_3 \pi$ taken in lines.

V. If the line p touches g_2, the two self-apolar sets of the fundamental involution of $R p \pi$ coincide, these being defined by the two points in which p meets g_2. The corresponding curve in π is $g_2 \pi$ taken in lines.

VI. The points of inflection of $R p \pi$ are defined on R by the six points in which p meets Σ. If the six points thus determined on R by p are a self-apolar sextic, the sextic giving the points of inflection of $R p \pi$ must be self-apolar. The corresponding curve in π is the locus of lines meeting $\Sigma \pi$ in six points whose parameters are a self-apolar sextic.

VII. If two points of inflection belong to one set of the fundamental involution, p meets γ_3. We shall see in § 9 that the conic on the points of inflection is then two lines. The corresponding curve in π is the product of the four points in which γ_3 meets π, the four nodes of $\Sigma \pi$.

VIII and IX will be proved in §§ 10 and 9 respectively.

§ 9.   *Syzygetic Lines and Planes and the Associated Hypercomplexes.*

Any point of F_4 determines a quartic on R. The Hessian of this quartic determines another point of F_4. If these two quartics are f and h, its Hessian, any point of the line joining them defines a quartic on R of the form

$$f + \lambda h = 0.$$

Since the Hessian of $f + \lambda h$ is of the form

$$f + \mu h = 0,$$

the line is equally well determined, and uniquely, by any one of its points. We shall call such a line in F_4 a *syzygetic line*, since its points give on R a syzygetic pencil of quartics. The totality of syzygetic lines we shall call *the system s*, and individual lines of the system will frequently be called *s-lines*. Polar planes of syzygetic lines as to R will be called *syzygetic planes*; we denote the system of these planes by the Greek letter σ. σ-planes may be defined independently as planes which are axes of pencils of spaces which cut out of R syzygetic pencils of quartics. There are ∞^4 points in F_4; each point, in general, deter-

mines uniquely an s-line, but since the line may equally well be determined by any of its points, we have:

The totality of s-lines in F_4 are a system of ∞^3 lines.

Also:

There are ∞^3 σ-planes in F_4.

There is a unique s-line on any point.

There is a unique σ-plane on any space.

In order to discuss s-lines and the associated theory it will be necessary to recall a few known facts from the invariant theory of the binary quartic.

Let a quartic f be written:

$$f \equiv a_0 t^4 + 4 a_1 t^3 + 6 a_2 t^2 + 4 a_3 t + a_4.$$

Its Hessian is:

$$h \equiv (a_0 a_4 - a_1^2) t^4 + 2 (a_0 a_3 - a_1 a_2) t^3 + (a_0 a_4 + 2 a_1 a_3 - 3 a_2^2) t^2$$
$$+ 2 (a_1 a_4 - a_2 a_3) t + a_2 a_4 - a_3^2.$$

Its cubicovariant is:

$$T \equiv (a_0^2 a_3 - 3 a_0 a_1 a_2 + 2 a_1^3) t^6 + (a_0^2 a_4 + 2 a_0 a_1 a_3 - 9 a_0 a_2^2 + 6 a_1^2 a_2) t^5$$
$$+ (5 a_0 a_1 a_4 - 15 a_0 a_2 a_3 + 10 a_1^2 a_3) t^4 + (- 10 a_0 a_3^2 + 10 a_1^2 a_4) t^3$$
$$+ (- 5 a_0 a_3 a_4 + 15 a_1 a_2 a_4 - 10 a_1 a_3^2) t^2$$
$$+ (- a_0 a_4^2 - 2 a_1 a_3 a_4 + 9 a_2^2 a_4 - 6 a_2 a_3^2) t + (- a_1 a_4^2 - 2 a_3^3 + 3 a_2 a_3 a_4).$$

Let f have a double root. This is equivalent to saying that the point defining f is on Σ. Suppose the double root to be $t = 0$. Then

$$f \equiv a_0 t^4 + 4 a_1 t^3 + 6 a_2 t^2,$$
$$h \equiv (a_0 a_2 - a_1^2) t^4 - a_2 (2 a_1 t^3 + 3 a_2 t^2).$$

In this case we have:

$$a_2 f + 2 h = (3 a_0 a_2 - 2 a_1^2) t^4. \tag{1}$$

The vanishing of the factor $3 a_0 a_2 - 2 a_1^2$ is the condition that the other two roots of f are equal; this is significant as we shall see shortly.

It follows from (1) that if f represents a point on an osculating plane of R, the s-line determined by f passes through the point of osculation of this plane with R. Hence:

Every line in an osculating plane of R and through the point of osculation is an s-line.

Again, let us suppose that f is given in the canonical form

$$f \equiv a_0 t^4 + 6 a_2 t^2 + a_4.$$

Then $a_1 = a_3 = 0$ and

$$h = a_0 a_2 t^4 + (a_0 a_4 - 3 a_2^2) t^2 + a_2 a_4,$$

$$T = (a_0 a_4 - 9 a_2^2)(a_0 t^6 - a_4 t).$$

Here

$$a_2 f - h = (9 a_2^2 - a_0 a_4) t^2. \tag{2}$$

The factor $9 a_2^2 - a_0 a_4$ on the right again represents by its vanishing the condition that f be the square of a quadratic.

T is the product of three quadratics, the Jacobians of the three pairs of quadratic factors of f. In our canonical form the roots of one of these quadratics are 0 and ∞. Equation (2) states the well-known fact that the square of this quadratic is in the pencil $f + \lambda h$; if $9 a_2^2 - a_0 a_4 = 0$, f and h are the same quartics and $T \equiv 0$. Now in the pencil $f + \lambda h$ occur the squares of the quadratic factors of T; any quartic which is a perfect square is represented in F_4 by a point on two osculating planes of R; in other words, a point on the double spread of Σ, γ_3. Hence:

Syzygetic lines in F_4 are lines trisecant to γ_3.

This may be used as a geometrical definition of s-lines, and is very important for our purpose.

If f is a perfect square, f and h are the same, and we have:

The points of γ_3 fail to determine s-lines uniquely.

This is the meaning, from our point of view, of the factors on the right in (1) and (2).

The three quadratic factors of the sextic T are apolar two and two. The theory of quadratics on R is closely associated with the spreads γ_3 and its dual g_3. A pair of points on R defines uniquely a point of γ_3, the point of intersection of the planes of R at the point. We have:

The necessary and sufficient condition that two quadratics on R be apolar is that the line joining the points of γ_3 representing them be an s-line; that is, meet γ_3 again.

This is an interpretation in four dimensions of the theorem that if two quadratics are apolar there is an identical relation among the squares of the quadratics and the square of their Jacobian.

We have pointed out that every space section of γ_3 is a rational quartic.

The s-lines in a, space α, that is, lines trisecant to the rational quartic $\gamma_3 α$, are therefore lines of a quadric $s.α$.

Since all lines in a plane of R and through the point of osculation are s-lines, we have:

The quadric $s.α$ contains the cuspidal lines of $Sα$.

Consider a point x of γ_3 and a space α on it. There is a unique line through x trisecant to $\gamma_3 α$. Hence:

s-lines on a point x of γ_3 are on a plane π_x.

The locus of pairs of points of γ_3 on an s-line through x is a conic c_x in the plane π_x.

For each s-line through x meets this locus in two and only two points.

The following properties of the conic c_x determined on γ_3 by a point x of γ_3 are evident from the theory of binary quadratics:

c_x meets R in two points, a_{t_1} and a_{t_2}.

The planes of R at t_1 and t_2 meet at x.

x is the polar point, on π_x, of the line $a_{t_1} a_{t_2}$ as to c_x.

If x' is a point of c_x, the conic $c_{x'}$ passes through x. There are thus ∞^1 conics $c_{x'}$ through x.

The correspondence between conics c_x on γ_3 and binary quadratics on R is one-to-one.

We are not concerned here primarily with the theory of the rational quartic curve in space. Many of its properties follow from the theory which we have developed. In this connection we refer the reader to the papers by Berzolari and Marletta previously mentioned.

We point out in passing the geometrical representation of the theory of the binary quartic which we have from our point of view. Any point x of F_4 determines a binary quartic on R,

$$f = (x\,t)^4 = 0.$$

The invariants of this quartic are represented by the spreads g_2 and g_3. Σ is the discriminant. The unique s-line, p_x, through x meets γ_3 in three points x_1, x_2, x_3, defining thus the three quadratics whose product is the sextic covariant of $(x\,t)^4 = 0$. Points of p_x represent quartics of the syzygetic pencil $f + \lambda\,h = 0$. The polar point (along p_x) of x as to x_1, x_2, x_3 gives the Hessian of f.

There are ∞^3 s-lines in F_4, and ∞^2 planes on each; we have thus ∞^5 planes. Hence:

Planes on s-lines in F_4 are in a hypercomplex of planes, Γ.

The planes $\Gamma.\alpha$ touch the quadric $s.\alpha$; hence:

Γ *is a quadratic hypercomplex.*

Dually we have:

Lines on σ-planes in F_4 are in a quadratic hypercomplex of lines, G.

The lines $\Gamma.a\alpha$ are a quadratic complex of lines on α. On α we have also the Steiner quartic surface $\gamma_3 a\alpha$. $Ra\alpha$ is an asymptotic curve on $\gamma_3 a\alpha$. The lines of $\Gamma.a\alpha$ may be regarded as the projections of s-lines from a on α; that is,

$$\Gamma . a\alpha \equiv s\,a\,\alpha.$$

Consider a point x of γ_3; we have a plane pencil of s-lines on x. $x\,a\,\alpha$ is a point of $\gamma_3\,a\,\alpha$. The plane pencil of s-lines on x projects into a plane pencil of lines of $\Gamma.a\alpha$ on $x\,a\,\alpha$. Hence:

$\gamma_3\,a\,\alpha$ *is the singular surface of the complex $\Gamma.a\,\alpha$.*

Stahl proved the existence of the complex $\Gamma.a\,\alpha$ from an entirely different point of view.[*] We omit the consideration of its properties for the sake of brevity.

Consider a line p in F_4; there is a unique s-line on every point of p. Planes on these s-lines and the line p generate the cone-spread $\Gamma.p$ determined by p, the locus of planes of Γ which contain p. p meets Σ in six points, determining six planes π_{t_i}, and hence six points a_{t_i}, of R. These six points project into the six points of inflection of $Rp\pi$. Let p meet π_{t_i} in the point b_i. Then, from what we have said, $b_i\,a_{t_i}$ is an s-line. Hence:

The conic $\Gamma.p\pi$ is the conic on the points of inflection of $Rp\pi$.

This conic may obviously be regarded as the projection from p of the points representing the Hessians of the various sets of the fundamental involution of $Rp\pi$ determined by the points of p. Algebraically, if we have a quartic given by

$$\rho X_i = (\alpha_i t)^4,\tag{3}$$

and its fundamental involution is

$$(at)^4 + \lambda\,(bt)^4 = 0,\tag{4}$$

and we represent by h the Hessian of the quartic (4) involving λ to the second

* Stahl, *Journal für Math.*, 101, p. 307.

degree, the conic on the six points of inflection of (3) is, λ being the variable parameter,

$$\rho X_i = |\alpha_i h|^4. \tag{5}$$

The points of the conic on the points of inflection of $Rp\pi$, which we shall call w, are thus put into one-one correspondence with the sets of the fundamental involution.

Let f be a point of p. Let the s-line, p_f, on f meet γ_3 in x_1, x_2, x_3. There are three planes on p_f which cut out conics from γ_3, the planes π_{x_1}, π_{x_2}, π_{x_3}, say. These three planes meet R in the quadratics defined by x_1, x_2, x_3,—three mutually apolar quadratics. Now any space on π_{x_1}, and in particular the unique space on π_{x_1} and p, touches γ_3 and cuts out harmonic pairs from R. We have then in π:

The three lines on a point of w which cut out from $Rp\pi$ harmonic pairs, meet $Rp\pi$ in three mutually apolar quadratics.

This is a defining characteristic of w.

If a plane π is on an s-line, $\Sigma\pi$ has three double points on a line, since s-lines are trisecant to γ_3. Hence our statement in IX, § 8.

s-lines meeting a line p are obviously on a two-way spread containing p. Any space α on p meets this two-way in p and in the two s-lines on the points in which p meets the quadric $s.\alpha$. Hence:

s-lines meeting a line p are on a cubic two-way, T_3,[*] with p for a directrix. This T_3 meets R in the six points determining the points of inflection of $Rp\pi$.

This T_3 meets γ_3 in a curve. Any space α on p meets this curve in six points, three on each s-line on α and meeting p. Hence:

s-lines which meet a line p meet γ_3 in a curve of order 6.

It will be obvious from the next section that the genus of a curve of order 6 on γ_3 can not be greater than one. The above curve projects from a point of F_4 into an elliptic sextic whose trisecant lines are lines of a ruled cubic surface.

The quadric spread $\Gamma.p$ meets γ_3 in a curve of order 8, out of which this sextic factors. $\Gamma.p$ meets γ_3 in an additional conic, C. Now C meets R in a pair of points defined by a point x common to γ_3 and the plane of C. C is the locus of points on γ_3 defined by pairs of points apolar to the points in which C meets R. $Cp\pi$ is the conic w of $Rp\pi$. w meets $Rp\pi$ in the six points of inflection and a further pair of points, which we shall call the points q. The

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[*] T_3 will be used as a generic symbol for the so-called normal cubic two-way in F_4.

points q are the projections of the points in which C meets R. We may sum this up in the theorem:

> *Line sections of $Rp\pi$ on any point of w are apolar to the square of a quadratic which is itself apolar to q. Lines on the polar point of the line joining the points q as to w cut out of $Rp\pi$ quartics apolar to the square of q.*

If the roots of q are $(mt) = 0$ and $(nt) = 0$, and we write

$$(pt)^2 \equiv (mt)^2 + \lambda(nt)^2,$$

then the conic w may be written parametrically

$$\rho\,x_i = |a_i p|^2\,|a_i p'|^2,$$

where the λ in $(pt)^2$ is regarded as the varying parameter.

§10. *The Mapping of γ_3 on a Cutting Plane π: the Stahl Conic N of $\Sigma\pi$.*

The quartic determined on R by a point x of F_4 is:

$$x_0 t^4 + 4 x_1 t^3 + 6 x_2 t^2 + 4 x_3 t + x_4 = 0. \tag{1}$$

If this is the square of a quadratic, say

$$(a_0 t^2 + 2 a_1 t + a_2)^2 = 0, \tag{2}$$

then x is a point of γ_3. Comparing (1) and (2), we have:

$$\left.\begin{aligned}
\rho\,x_0 &= a_0^2, \\
\rho\,x_1 &= a_0 a_1, \\
\rho\,x_2 &= \tfrac{1}{3}(a_0 a_2 + 2 a_1^2), \\
\rho\,x_3 &= a_1 a_2, \\
\rho\,x_4 &= a_2^2.
\end{aligned}\right\} \tag{3}$$

If a_0, a_1 and a_2 are regarded as the homogeneous coordinates of a point in a plane π, the equations (3) define a mapping of π on γ_3 by means of a fourfold system of conics. If a_0, a_1, a_2 are line coordinates in π, we may define the system of conics

$$(\xi x) = 0$$

as all conics in π apolar to the conic

$$4 a_0 a_2 - a_1^2 = 0. \tag{4}$$

We shall call (4) the conic N in π. In a more geometrical mapping of π

on γ_3 to be mentioned shortly, this conic will be shown to be identical with the Stahl conic N of $\Sigma \pi$.*

N may be given parametrically:

$$a_0 = 1, \quad a_1 = -t, \quad a_2 = t^2. \tag{5}$$

The mapping defined by (3) *has no critical points; it follows that there exist no curves of odd order on* γ_3; *any curve of even order* $2p$ *on* γ_3 *is the map of a curve of order* p *in* π.

From (3) and (5) we have:

N *maps into the quartic* R.

Lines of π pass into conics on γ_3. The plane on this conic meets γ_3 again in the map of the polar point of the given line as to N. We have thus the ∞^2 conics c_x determined by points x of γ_3 referred to in the previous section. Spaces on the plane of a conic c_x cut out of γ_3 a varying conic $c_{x'}$; x' is on c_x. $c_{x'}$ is the map of a line on π on the polar point as to N of the line representing c_x.

The tangents to N map into conics on osculating planes of R; these conics obviously touch R at the point of osculation. *They are in fact the pure*† *second osculants of* R. For the second osculant at a point t of R is cut out of the developable of the first osculant by the plane of the osculant at t. This plane is also the plane of R at t. It follows at once that

γ_3 *is the locus of pure second osculants of* R.

The polar point of a line of N as to N is its point of contact. It follows that any plane of R touches γ_3 at its point of osculation, and we see again that any line in an osculating plane of R and through the point of osculation is to be regarded as an s-line. Any line in a plane of R is a bisecant line to γ_3.

The quadratic defined by a point x of γ_3 is given on N by tangents to N from the map of x on π.

Space sections of γ_3 are represented on π by conics apolar to N. Points of intersection with γ_3 of s-lines are sets of three points on π which carry a *double*

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* N is defined as the locus of the line through the three points of inflection of a cubic osculant of a quartic R p π. Stahl, *loc. cit.*, p. 314.

† If a rational curve is given parametrically,

$$\rho x_i = (a_i t)^n,$$

we call the curves

$$\rho x = (a_i t_1)^2 (a_i t)^{n-2}$$

pure second osculants;

$$\rho x_i = (a_i t_1)(a_i t_2)(a_i t)^{n-2}$$

are *mixed* second osculants.

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infinity of conics apolar to N; that is, by three-points apolar to N. The ∞^1 s-lines on a space section of γ_3 are represented by the ∞^1 triangles, inscribed in a conic apolar to N, and apolar to N.

The points in which any plane π' meets γ_3 are the maps of the common points on π of a pencil of conics all of which are apolar to N. Such a set of four points will be said to be *orthic* to N. This is the idea of the "Polviereck" of Reye. It follows that

The parameters of the nodes of $\Sigma \pi'$ are represented on N by the tangents from points of a four-point orthic to N.[*]

Such a set of quadratics will be called an *orthic* set.

If three quadratics are the squares of the roots of a cubic equation, the unique quadratic making up the orthic set is known to be the Hessian of the cubic. A plane trisecant to R may be considered as representing a cubic on R. Its Hessian is given by the fourth point x in which this plane meets γ_3. Its system of polar quadratics are defined by points of the unique conic on γ_3 whose plane passes through x.

Consider in F_4 a space α and a plane π on it. Let α cut out of R the points a_{t_i} with parameters t_i. We call, as before, the lines and planes of R at these points respectively p_{t_i} and π_{t_i}.

$\gamma_3 \alpha$ is a rational quartic in α. $\Sigma \alpha$ is the developable of $S\alpha$ and $\gamma_3 \alpha$ is its double curve. $s \cdot \alpha$ is the unique quadric on $\gamma_3 \alpha$. $s \cdot \alpha$ contains the cusp tangents of $S\alpha$; that is, the lines $\pi_{t_i} \alpha$. Now, as we have said, the parameters t_i are a set of the fundamental involution on $\Sigma \pi$. $s \cdot \alpha \pi$ is a conic on π, on the nodes of $\Sigma \pi$ and cutting out from $\Sigma \pi$ an additional set of four points with the parameters t_i. Hence:

Conics on the nodes of $\Sigma \pi$ cut out of $\Sigma \pi$ sets of the fundamental involution on $\Sigma \pi$.[†]

The quadric $s \cdot \alpha$ contains two systems of generators. Let us call s-lines generators of the *first* system of $s \cdot \alpha$, and lines on this quadric meeting s-lines generators of the *second* system.

Projecting $S\alpha$ from a cusp a_{t_i}, we obtain, in accordance with a former section, a cubic osculant of $\Sigma \pi$, the osculant at the point t_i. Let us call this osculant Δ_i. The three cusp tangents of Δ_i are the projections from a_{t_i} of the lines $\pi_{t_k} \alpha$, $\pi_{t_l} \alpha$, $\pi_{t_m} \alpha$. The point in which these cusp tangents meet is the inter-

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* Meyer: "Apolarität," pp. 241, ff.

† This is essentially Stahl's proof of this fact: *Journal für Math.*, 101, p. 302.

section with π of the generator of the second system of $s.\alpha$ through a_{t_i}. The locus of such points for all spaces α on π is the Stahl conic N of $\Sigma\pi$. We have at once:

A conic on the nodes of $\Sigma\pi$ cuts out of the Stahl conic N of $\Sigma\pi$ the same set of the fundamental involution of $\Sigma\pi$ which it cuts out of $\Sigma\pi$.

The section, $\gamma_3\alpha$, of γ_3 by any space α on π is uniformly represented on the conic $s.\alpha\pi$, the conic in which the unique quadric on $\gamma_3\alpha$ meets π, by means of the generators of the *second* system on $s.\alpha$, these generators meeting $\gamma_3\alpha$ once. We shall see that in this way γ_3 is mapped in a one-to-one way on the plane π.

Any point x of γ_3 determines a space α on x and π, and thereby a quadric $s.\alpha$ on α, and a unique point on $s.\alpha\pi$, the generator of the second kind on $s.\alpha$ on x meeting π in this point. The conics $s.\alpha\pi$ are in a pencil on π, their common points of intersection being the four nodes of $\Sigma\pi$. Now there can not be two quadrics $s.\alpha$ and $s.\alpha'$ on one conic of this pencil in π since this would require two s-lines through every point of this conic, and we know that there is a unique s-line on every point of F_4 not on γ_3. We have then:

The correspondence between γ_3 and π determined by the quadrics $s.\alpha$ on spaces α on π is one-to-one. Conics of the pencil on the four points $\gamma_3\pi$ correspond to the space sections $\gamma_3\alpha$ on π.

Further, there are no singular points of this correspondence in π; that is, there are no points of π which have no definite correspondents on γ_3. Now the space sections of γ_3 on π are represented by conics; it follows that *all* space sections of γ_3 are maps of conics on π. Hence we have:

γ_3 is mapped by our scheme from π by conics of a fourfold system; that is, by conics apolar to a definite conic, C, on π.

Let us call the pencil of conics on the nodes of $\Sigma\pi$ in π the pencil Ψ. On any conic of Ψ there is a natural three-to-one correspondence. For on any point of such a conic there is a unique s-line, p; on this s-line and π there is a space α. p meets $\gamma_3\alpha$ in three points; these three points give in turn three points on the conic of Ψ, $s.\alpha\pi$. These three points are, from what we have said in the first part of this section, apolar to the conic C of the above theorem. C is the locus of points in which two of such sets of three points on conics of the pencil Ψ coincide. Any space α on π meets R in four points a_{t_i}; the lines $\pi_{t_i}\alpha$ are s-lines on $s.\alpha$. *But these lines touch γ_3 and hence $\gamma_3\alpha$.* It follows that the generator of the second system of $s.\alpha$ on a_{t_i} meets π in a point of the conic C. Hence:

The conic C on π is the Stahl conic N of $\Sigma\pi$.

Four points on N determine four points on R. These in turn determine a space α in F_4. The quartic $\gamma_3 \alpha$ is the map of the unique conic on the four points of N and apolar to N. The tangents at the four points of N determine a range of conics. Conics apolar to all conics of this range are the maps of space sections of γ_3 on the polar point a of α as to g_2. We have thus the mapping from π of the Steiner surface $\gamma_3 a \alpha'$. We point out in passing that it follows easily from this that *the involution $I_{2,2}$ determined on $\Sigma \pi$ by tangents from a point of π is defined on N by points x, y, conjugate as to the pencil Ψ.*

Let us call the generators of the second system passing through a_{t_i} of the quadric $s.\alpha$ on a space α on π, the lines l_1, l_2, l_3, l_4. l_1 is on a_{t_1} and meets $\pi_{t_2}\alpha$, $\pi_{t_3}\alpha$, $\pi_{t_4}\alpha$. π_{t_1}, π_{t_2}, π_{t_3}, π_{t_4} meet π in four points, say b_1, b_2, b_3, b_4 of the fundamental involution of $\Sigma \pi$ on $\Sigma \pi$. l_1, l_2, l_3, l_4 meet π in four points of the same set of the fundamental involution of $\Sigma \pi$ on N. Call these the points a_1, a_2, a_3, a_4. The plane $\pi_{t_1}\alpha l_2$ meets π in the line $b_1 a_2$.

The projection of $S\alpha$ on π from a_{t_1} is the first osculant Δ_1 of $\Sigma \pi$ at the point b_1. The cusps of Δ_1 are the projections of a_{t_2}, a_{t_3}, a_{t_4}, and its cusp lines are the projections of $\pi_{t_2}\alpha$, $\pi_{t_3}\alpha$, $\pi_{t_4}\alpha$.

The line $b_1 a_2$ in π is the cusp tangent 1 to the osculant Δ_2. The lines $b_1 a_2$ and $b_2 a_1$ meet in the point 12 of $g_3 \alpha$, the trace of $a_{t_1} a_{t_2}$ on α. The two sets of four points a_1, a_2, a_3, a_4, b_1, b_2, b_3, b_4 are on a conic Ψ on the nodes of $\Sigma \pi$. The three points $b_1 a_2/b_2 a_1$, $b_2 a_3/b_3 a_2$, $b_3 a_1/b_1 a_3$ are on a line by the Pascal theorem. This is the line 4 of the Stahl conic K of $\Sigma \pi$; it is the trace on π of the plane $a_{t_1} a_{t_2} a_{t_3}$. We may summarize what we have said in the following theorem:

A conic Ψ on the nodes of $\Sigma \pi$ cuts out of $\Sigma \pi$, and out of its Stahl conic N, a set of the fundamental $I_{1,3}$ of $\Sigma \pi$; we have thus on $\Sigma \pi$ four points b_i and on N four points a_i, these two sets of four points being ordered with respect to each other. Points of intersection of pairs of lines like $b_1 a_2/a_1 b_2$ — six points in all — are six vertices of a four-line inscribed in the cubic $g_3 \pi$. The lines of this four-line touch the Stahl conic K of $\Sigma \pi$ and define on K the set of the fundamental $I_{1,3}$ of $\Sigma \pi$ which Ψ cuts out of $\Sigma \pi$ and N.

If p is a line of F_4 meeting g_3 in a point c, and the bisecant line to R through c meets R in d_{t_1} and d_{t_2}, and the plane of R at d_{t_3}, say, meets p, then $Rp\pi'$ is a rational quartic with a flecnode.

The line $b_1 a_2$ on the plane π above is such a line. But $b_1 a_2$ is a cusp-line of a cubic osculant of $\Sigma \pi$. Hence the statement in IX, § 8.

§ 11. *Three-to-one Correspondence Determined on π by $\Sigma \pi$ and its Stahl conic N.*

It is not possible to establish a one-one correspondence between a conic in a plane and a sextic with six cusps and four nodes by means of a Cremona transformation of the plane. We have suggested by the preceding section a three-to-one correspondence in the plane π which relates N with $\Sigma \pi$ in a one-to-one way.

Spaces α on π, as we have seen, meet γ_3 in rational quartics $\gamma_3 \alpha$; the quadrics $s.\alpha$ containing the quartics $\gamma_3 \alpha$ meet π in conics of a pencil Ψ on the nodes a_1, a_2, a_3, a_4, say, of $\Sigma \pi$. The four points a_i are an orthic set of four points as to the conic N of $\Sigma \pi$.

Through any point x of π there is a unique s-line; there is a unique space α on this s-line and the plane π. α contains a quadric $s.\alpha$ which meets π in a conic Ψ on a_1, a_2, a_3, a_4, x. The s-line on x lies on this quadric. This s-line meets the curve $\gamma_3 \alpha$ in three points. We call, as before, s-lines generators of the first system on $\gamma_3 \alpha$, and lines on $\gamma_3 \alpha$ which are not s-lines, generators of the second system. The three generators of the second system of $s.\alpha$ on the three points in which the s-line on x meets $\gamma_3 \alpha$, meet π in three points on the conic Ψ and apolar to N. For any point x of π we determine in this way three points, y_1, y_2, y_3, on the same conic Ψ with x. There is thus determined a three-to-one correspondence of points y and x of the plane π; given a point y we have one x, for y determines a conic Ψ which in turn determines a unique space α on π; the generator of the second system on $s.\alpha$ meets $\gamma_3 \alpha$ once; the s-line on $\gamma_3 \alpha$ on this point meets π in the point x. Conversely, given a point x, we have three points y. This may be regarded as a three-to-one correspondence in the whole plane π or as a three-to-one correspondence on a conic Ψ. Let us examine the correspondence first from the latter point of view.

Looked upon as determined by the values of a binary parameter on Ψ the sets of points y — three-points inscribed in the conic Ψ and apolar to N — are represented by a pencil of binary cubics

$$(\alpha t)^3 + \lambda (\beta t)^3 = 0. \tag{1}$$

Such a pencil of cubics is known to be the polar cubics of a unique quartic f, and the apolar cubics of a second quartic $\bar{f}$. f and $\bar{f}$ have the same Hessian, this Hessian being the four points in which two points of a cubic of the set (1) coincide. The common Hessian of f and $\bar{f}$ is therefore given by the points of intersection of Ψ and N. We write the system of points y_i in the form

$$(f t)^3 (f t') = 0, \tag{2}$$

(2) being the polarized form of the quartic

$$(f t)^4 = 0.$$

The most general correspondence between the groups of three points (2) and a point τ of Ψ may be obtained by putting in (2)

$$t' = \frac{a\tau + b}{c\tau + d},$$

where a, b, c, d are arbitrary constants. If we take f in the canonical form

$$(ft)^4 \equiv f_0 t^4 + 6 f_2 t + f_4 = 0,$$

we have

$$(ft)^3 (ft') \equiv t' (f_0 t^3 + 3 f_2 t) + 3 f_2 t^2 + f_4 = 0, \qquad (3)$$

and putting

$$t' = \frac{a\tau + b}{c\tau + d},$$

(4) becomes, after putting $\tau = t$,

$$a f_0 t^4 + (b f_0 + 3 c f_2) t^3 + (3 a f_2 + 3 d f_2) t^2 + 3 (b f_2 + c f_4) t + d f_4 = 0.$$

(4) gives the points of coincidence ($\tau = t$) of our new correspondence. These four points, as is obvious, can not be assigned arbitrarily; we show that they must be apolar to $\bar{f}$, and hence must be orthic to N.[*] If t_1, t_2, t_3 and t_4 are the four coincidence-points of our correspondence on Ψ, and s_1, s_2, s_3 and s_4 the elementary symmetric functions of the four t's, we have, comparing the equation giving the four t's with (4), and eliminating a, b, c and d,

$$\begin{vmatrix} f_0, & 0, & 0, & 0, & 1 \\ 0, & f_0, & 3 f_2, & 0, & -s_1 \\ 3 f_2, & 0, & 0, & 3 f_2, & s_2 \\ 0, & 3 f_2, & f_4, & 0, & -s_3 \\ 0, & 0, & 0, & f_4, & s_4 \end{vmatrix} = 0,$$

or

$$(f_0 f_4 - 9 f_2^2) [3 f_0 f_2 s_4 - f_0 f_4 s_2 + 3 f_2 f_4] = 0. \qquad (5)$$

It is easily verified that (5) is the completely polarized form of the quartic $\bar{f}$. Hence:

   Given any three-to-one binary correspondence of which the sets of three are polar cubics of a binary quartic f, the coincidence-points of the correspondence are apolar to the quartic $\bar{f}$ having the same Hessian as f.

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[*] Grace and Young: *Algebra of Invariants*, p. 310.

Let us call the quartic determined in this way by a line l_y the quartic $Q_y^{(i)}$. There is a unique curve $Q_y^{(i)}$ with a node at a given point. There are thus ∞^2 such curves, corresponding to the ∞^2 lines of π, when B is given.

d) *The direction of coincidence of two points of a set $\iota_3^{(3)}$ at a point of N defines a rational curve S' of class 6.* This curve has for double tangents the lines of B. It is the polar reciprocal as to N of the rational sextic mentioned above with nodes at the points of B.

A curve C_x must be rational since it is in one-one correspondence with a line l_y. We now locate its node. Let n_1 be the pole of l_y as to N. The conic on a_1, a_2, a_3, a_4, n_1 cuts out of l_y two points n_2 and n_3 such that n_1, n_2, n_3 are a group of y-points, a set $\iota_3^{(3)}$. To these corresponds one point, m, on this conic. m is the double point of C_x, the two points n_2 and n_3 of l_y giving the one point m.

Two curves C_x and C_x' meet in five variable points — four points of intersection of these cubics being fixed in the points a_i. One of these five points is the transform of the point in which l_y meets l_y'. The other four points of intersection of C_x and C_x' arise from pairs of points $y_1 y_2$, which fall respectively on l and l'. As a point y_1 runs along l, the locus of the points y_2, y_3 is a quartic curve with a node at the pole of l as to N. This curve, $Q_y^{(i)}$, meets l' in four points, thus determining four pairs $y_1^{(i)} y_2^{(i)}$ on l and l', and thence the four further intersections of C_x and C_x'.

To a point y corresponds in general a unique x; to the point $p_1 p_2$ correspond both a_1 and a_2; it follows that $p_1 p_2$ is a singular point of $T\begin{pmatrix} 3 & 1 \\ y, & x \end{pmatrix}$, and that the corresponding locus of x's passes through a_1 and a_2. This locus can be nothing but the line $a_1 a_2$. $T\begin{pmatrix} 3 & 1 \\ y, & x \end{pmatrix}$ has therefore the six singular points $p_1 p_2$, etc., with the six corresponding lines $a_1 a_2$, etc. It follows that $T\begin{pmatrix} 1 & 3 \\ x, & y \end{pmatrix}$, *in general, sends lines into cubic curves on the six points $p_i p_j$, the six points of the four-line p_i. The curves C_y are a net of cubics on these six points.*

There is a unique pair of points y_1, y_2 belonging to the same set $\iota_3^{(3)}$ on any line l_y of the plane. We saw that this pair of points y transform into the node of C_x. If C_x is to have a cusp, the pair of points y_1, y_2 on the line l_y must coincide. Hence:

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The locus of lines l_y transforming by $T \begin{pmatrix} 3 & 1 \\ y, & x \end{pmatrix}$ into cubic curves C_x with a cusp is the sextic line-curve S' of theorem d).

It follows immediately:

The locus of cusps of cuspidal curves C_x is the transform of N by $T \begin{pmatrix} 3 & 1 \\ y, & x \end{pmatrix}$; that is, the sextic $\Sigma \pi$.

$T \begin{pmatrix} 3 & 1 \\ y, & x \end{pmatrix}$ may easily be put into analytical form. Let us take as a_1, a_2, a_3, a_4 the four points $(1, 0, 0), (0, 1, 0), (0, 0, 1), (1, 1, 1)$ respectively. Then N, in lines, must be of the form

$$k_0 \xi_0^2 + k_1 \xi_1^2 + k_2 \xi_2^2 + 2\lambda (\xi_1 \xi_2 + \xi_2 \xi_0 + \xi_0 \xi_1) = 0.$$

N is, therefore, in points,

$$N \equiv (k_1 k_2 - \lambda^2) x_0^2 + (k_2 k_0 - \lambda^2) x_1^2 + (k_0 k_1 - \lambda^2) x_2^2$$
$$+ 2\lambda (\lambda - k_0) x_1 x_2 + 2\lambda (\lambda - k_1) x_2 x_0 + 2\lambda (\lambda - k_2) x_0 x_1 = 0.$$

Then

$$p_1 \equiv (k_1 k_2 - \lambda^2) y_0 + \lambda (\lambda - k_2) y_1 + \lambda (\lambda - k_1) y_2,$$
$$p_2 \equiv \lambda (\lambda - k_2) y_0 + (k_2 k_0 - \lambda^2) y_1 + \lambda (\lambda - k_0) y_2,$$
$$p_3 \equiv \lambda (\lambda - k_1) y_0 + \lambda (\lambda - k_0) y_1 + (k_0 k_1 - \lambda^2) y_2,$$
$$p_4 \equiv p_1 + p_2 + p_3$$
$$\equiv (\lambda - k_1)(\lambda - k_2) y_0 + (\lambda - k_2)(\lambda - k_0) y_1 + (\lambda - k_0)(\lambda - k_1) y_2.$$

$T \begin{pmatrix} 3 & 1 \\ y, & x \end{pmatrix}$ is at once

$$x_0 = (\lambda - k_1)(\lambda - k_2) y_0 p_2 p_3,$$
$$x_1 = (\lambda - k_2)(\lambda - k_0) y_1 p_3 p_1,$$
$$x_2 = (\lambda - k_0)(\lambda - k_1) y_2 p_1 p_2.$$

The curves C_y are a net of cubic curves on the six-points $p_i p_j$. These curves admit an infinity of point-sets $\iota_8^{(3)}$, and hence meet N with the direction of the tangent to the sextic S' of theorem d) determined by a point of inter-section with N. It follows that N is the curve along which curves of the net C_y touch; that is,

N is the Jacobian curve of the net of curves C_y.

The Jacobian curve of a net of curves is also the locus of nodes of nodal curves of the net; hence:

There is a unique curve C_y with a node at a given point of N.

Our whole apparatus is obviously determinable in the reverse order. That is, we may start with four lines p_1, p_2, p_3, p_4, and a net of cubic curves C_y on the six points $p_i p_j$, the points of intersection of these four lines. The Jacobian curve of the net C_y is the four lines p_i and a conic N orthic to the four lines. Any three points which are the base-points of a pencil of the net C_y are a set $\iota_3^{(3)}$ as determined by the conic N and the orthic four-line p_1, p_2, p_3, p_4. This apparatus has eleven constants, as it should.

Consider a line l_x and its transform C_y. The six points in which C_y meets N are transforms of the six points in which l_x meets $\Sigma \pi$. Now all curves C_y have the same tangent at a point of N; if we ask for a curve C_y meeting N in a point counting twice, C_y must have a node at this point. The corresponding line l_x must meet $\Sigma \pi$ in a point counting twice; that is, must touch $\Sigma \pi$. Hence:

Curves C_y with a node are transforms of lines of $\Sigma \pi$.

Any locus of sets $\iota_3^{(3)}$ is sent by $T\begin{pmatrix} 3 & 1 \\ y, & x \end{pmatrix}$ into some curve repeated three times. Conversely, to a general curve corresponds by $T\begin{pmatrix} 1 & 3 \\ x, & y \end{pmatrix}$ a locus of sets $\iota_3^{(3)}$.

$T\begin{pmatrix} 3 & 1 \\ y, & x \end{pmatrix}$ *sends a curve C_y into its correspondent line l_x taken three times. It sends conics on sets of four points like a_4, $p_2 p_3$, $p_3 p_1$, $p_1 p_2$,—another orthic four-point of B,— into lines on a_4.*

The latter system of conics taken with p_4 are special curves C_y.

$T\begin{pmatrix} 3 & 1 \\ y, & x \end{pmatrix}$ *sends cubics of a net on six points of B like a_1, a_2, a_3, $p_1 p_4$, $p_2 p_4$, $p_3 p_4$,— vertices of another orthic four-line of B,— into conics on a_1, a_2 and a_3.*

· This net of cubics is a net similarly associated with another correspondence determined by the orthic four-line $a_1 a_2$, $a_2 a_3$, $a_3 a_1$, p_4.

There are *five* orthic four-points among the ten points of our configuration B, each determining a three-to-one correspondence of points of the plane π. To deal with these five correspondences it is desirable to make a change in our notation, conforming it with that of the former paper already referred to. A configuration B, as we have said, is cut by a plane out of a complete five-point in space, and a symmetrical notation for its points and lines is given by naming the points of

space 1, 2, 3, 4, 5, say, and using two-figure symbols for the points, and three-figure symbols for the lines of B. The following table indicates the desired change in the notation:

Points.		Lines.	
a_1	12	p_1	345
a_2	13	p_2	245
a_3	14	p_3	235
a_4	15	p_4	234
$p_1 p_2$	25	$a_1 a_2$	134
$p_2 p_3$	35	$a_2 a_3$	124
$p_3 p_1$	45	$a_3 a_1$	123
$p_1 p_4$	34	$a_1 a_4$	125
$p_2 p_4$	42	$a_2 a_4$	135
$p_3 p_4$	23	$a_3 a_4$	145

Any group of four points of B whose symbols contain the same figure, as 12, 13, 14, 15, is an orthic four-point; and the group of four lines from whose symbols the given figure is absent, as 345, 245, 235, 234, is the corresponding orthic four-line. We name the five orthic four-points of B the four-points 1, 2, 3, 4, 5, these giving the correspondences T_1, T_2, T_3, T_4, T_5 respectively.

The sets y are the same for all correspondences T_i. They are sets $\iota_3^{(3)}$ as determined by the configuration. We have thus five nets of cubics $C_y^{(i)}$; one net on the six points of each of the orthic four-lines of B. The characteristic feature of the cubics $C_y^{(i)}$ is that any curve of one of the nets admits an infinity of sets $\iota_3^{(3)}$. The curves $C_y^{(i)}$ correspond to lines in the transformation $T_i \begin{pmatrix} 1 & 3 \\ x, & y \end{pmatrix}$. N is the common Jacobian of all of these nets. Two cubics of the same net meet in three variable points — a set $\iota_3^{(3)}$; two cubics from different nets meet in *two* sets $\iota_3^{(3)}$.

To a given group, y_1, y_2, y_3, correspond by $T_i \begin{pmatrix} 3 & 1 \\ y, & x \end{pmatrix}$ five points, x_1, x_2, x_3, x_4, x_5. The correspondence between any two of these points, say x_1 and x_5, is one-to-one. x_1 determines by $T_i \begin{pmatrix} 1 & 3 \\ x, & y \end{pmatrix}$ a unique set $\iota_3^{(3)}$ which in turn, by $T_5 \begin{pmatrix} 3 & 1 \\ y, & x \end{pmatrix}$, determines a unique point x_5. Similarly, x_1 is unique when x_5 is given. Hence between the points x_1 and x_5 we have a Cremona correspondence;

there is a Cremona transformation T_{15} which sends x_1 into x_5, its inverse T_{51} sending x_5 into x_1.

T_{15} is the product of $T_1\begin{pmatrix}1 & 3\\x, & y\end{pmatrix}$ and $T_5\begin{pmatrix}3 & 1\\y, & x\end{pmatrix}$; thus

$$T_{15} = T_1\begin{pmatrix}1 & 3\\x, & y\end{pmatrix} T_5\begin{pmatrix}3 & 1\\y, & x\end{pmatrix}.$$

Now $T_1\begin{pmatrix}1 & 3\\x, & y\end{pmatrix}$ sends a line l_x into a cubic curve $C_y^{(1)}$. $T_5\begin{pmatrix}3 & 1\\y, & x\end{pmatrix}$ sends a cubic $C_y^{(1)}$ into a conic on 25, 35, 45. Hence T_{15} sends the line l_x into a conic on 25, 35, 45. Similarly T_{51} sends a line into a conic on 12, 13, 14. 15 is a fixed point for the correspondences T_1 and T_5, and hence is a fixed point for T_{15}. Hence:

T_{15} *is a quadratic Cremona transformation having* 12, 13, 14 *as singular points, with* 25, 35, 45 *as singular points of the inverse transformation,* T_{51}. *The transformation is completely defined by these facts and the additional condition that* 15 *be a fixed point.*

When the singular triangles and a pair of corresponding points are given for a quadratic Cremona transformation, a linear construction for the correspondence is known. This gives a geometrical determination of our correspondence $T_1\begin{pmatrix}1 & 3\\x, & y\end{pmatrix}$. Choose any other correspondence, say T_5. Let x' be the transform of x by T_{15}. The conic on x', 15, 25, 35, 45, meets the conic on x, 15, 12, 13, 14, in the point 15 and in a set $\iota_3^{(3)}$ which corresponds to x and x' in $T_1\begin{pmatrix}1 & 3\\x, & y\end{pmatrix}$ and in $T_5\begin{pmatrix}1 & 3\\x, & y\end{pmatrix}$ respectively. The linear construction of x when a corresponding y is given seems too complicated to be of much interest.

§ 12. *Further Theory of Quadratics on* R. *Curves Determined by a Quadratic on* $R\,p\,\pi$.

We saw in § 9 that the theory of quadratics on R is closely associated with that of the spread γ_3; dually we should expect the spread g_3 to be closely connected with quadratics on R. We shall point out in this section just what this relation is, and use the theory thus developed to show the existence of certain curves determined by quadratics on $R\,p\,\pi$. In the following section we shall use the same facts in the discussion of the nodes of $R\,p\,\pi$.

Any quartic on R determines a point in F_4. This is projected from a line p into a point in a plane π; this point is covariantly associated with the corre-

sponding quartic on $Rp\pi$. The determination of this covariant point obviously fails when the given quartic on R is a set of the fundamental involution of $Rp\pi$.

Conversely, given a point in π, there is a plane on p and this point, and hence there is an involution of ∞^2 quartics on R with which a point of π is associated in this manner. We have seen the way in which the coordinates of the point in π are determined when the binary quartic on R is given; if $Rp\pi$ is given by

$$\rho\, x_i = (\alpha_i t)^4, \tag{1}$$

and the given binary quartic is

$$(\mu t)^4 = 0,$$

then the point is

$$\rho\, x_i = |\alpha_i \mu|^4, \tag{2}$$

the definition of the point in π being merely that the sections of $Rp\pi$ by lines on it are apolar to the quartic μ. If

$$(f t)^4 + \lambda (f' t)^4 = 0$$

is the fundamental involution of $Rp\pi$, then (2) is equally well defined by any quartic of the system

$$(\mu t)^4 + \lambda (f t)^4 + \lambda' (f' t)^4 = 0.$$

If the given quartic $(\mu t)^4 = 0$ is the square of a quadratic, the point in F_4 which it defines is on γ_3. We have seen that the locus of points on γ_3 which define quadratics apolar to a given quadratic

$$(k t)^2 = 0 \tag{3}$$

is a conic, C_k, which meets R in the pair of points (3) and whose plane passes through the point on γ_3 defining (3), the point k say. Tangents from k to C_k touch at the points (3) of R. Let the pencil of quadratics apolar to $(k t)^2 = 0$ be

$$(p t)^2 = (a t)^2 + \lambda (b t)^2. \tag{4}$$

The coefficients of $(p t)^2$ involve the parameter λ linearly. From (4) we have

$$(k t)^2 \equiv |a\, b|\, (a t)(b t).$$

C_k projects from p into the conic $C_k p\pi$, on π. $C_k p\pi$ is, in terms of the parameter λ of (4),

$$\rho\, x_i = |\alpha_i p|^2 |\alpha_i p'|^2. \tag{5}$$

(5) is a conic on π covariantly associated with the quadratic $(k t)^2 = 0$ on $Rp\pi$.

It meets $R\,p\,\pi$ in the pair of points $(k\,t)^2 = 0$. The pole of the line joining this pair of points as to $C_k\,p\,\pi$ is the point

$$\rho\, x_i = |\,\alpha_i\, k\,|^2\,|\,\alpha_i\, k'\,|^2,$$

the point of π such that sections of $R\,p\,\pi$ through it are apolar to the square of the quadratic k.

There exist on the cubic spread g_3 a system of ∞^2 lines which are not bisecant to R, and which we may call after Segre, *axes* of g_3. There are three axes of g_3 on any space α; $g_3\,\alpha$ is a cubic surface with the four nodes $R\,\alpha$; it contains the six lines joining the four nodes, and three others lying on a plane, the unique σ-plane on α. The latter lines are axes of g_3.

Bisecant lines to R meeting an axis $p\,g$ of g_3 are on a two-way T_3, — of order 3, since this two-way is met by any space on the given axis $p\,g$ in $p\,g$ itself and in a pair of lines bisecant to R.[*] R meets the generators of this T_3 twice, and does not meet its directrix $p\,g$. The lines of T_3 meet R in pairs of an involution on R, or in quadratics apolar to a given quadratic on R, this quadratic being determined by a pair of tangents to R meeting $p\,g$. There is thus a unique axis of g_3, $p\,g$, meeting two tangent lines of R, — two lines of the spread of tangents S, — and the bisecant lines to R meeting $p\,g$ define on R the pencil of quadratics apolar to the pair of points of tangency of the two tangent lines. A space bitangent to R meet g_3 in a ruled cubic surface with the line joining the pair of points of tangency as a double line; the axis thus determined is the directrix of this ruled surface.

An interesting configuration of lines is determined on g_3 by three mutually apolar quadratics on R. Any point, a_1, of g_3 determines, by the bisecant line to R through it, a quadratic q_1 on R. Through a_1 there pass two axes of g_3, p_2 and p_3. Bisecant lines to R meeting p_2 and p_3 define on R the involutions of quadratics apolar to two further quadratics, q_2 and q_3. In particular, q_1 is apolar to q_2 and q_3. On the plane $p_2\,p_3$ there is a third axis of g_3, p_1. p_1, p_2 and p_3, three axes of g_3 on a σ-plane, meet in three points, a_1, a_2, a_3. These define by their bisecants to R three mutually apolar quadratics q_1, q_2, q_3. It is evident from what we have said that tangents to R at the two points q_1 meet a_1, etc.

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*[*] Cf. Segre, *loc. cit.*

The spread, Σ, of osculating planes of R, is a linear combination of g_2^3 and g_3^2, and hence can meet g_3 only in the spread

$$g_2 = g_3 = 0, \tag{6}$$

taken three times. Now (6) is the spread S of tangents to R. Hence:

$g_3\pi$ passes through the cusps of $\Sigma\pi$ with the cusp tangents of $\Sigma\pi$.

If $g_3\pi$ is three lines, π is a σ-plane, and we have:

The necessary and sufficient condition that $\Sigma\pi$ be a quartic projectively dual to the lemniscate is: a) $g_3\pi$ is three lines; or b) The fundamental involution of $\Sigma\pi$ is of the form $f + \lambda h = 0$, — a syzygetic pencil of quartics.

Dually:

R projects from an s-line, p, into a quartic curve $Rp\pi$ with three biflecnodes.

If π is on an axis of g_3, the curve $\Sigma\pi$ has a singularity which is the dual of a biflecnode. $g_3\pi$ is then a line and a conic. If π contains another axis of g_3, it must contain two. Hence, dually:

If a quartic $Rp\pi$ have a biflecnode and an additional flecnode, it must have three biflecnodes; that is, it must be a projection, real or imaginary, of the lemniscate.

All planes on an s-line are planes of the hypercomplex Γ of § 9. Hence:

The conic, w, on the points of inflection of a quartic with three biflecnodes vanishes identically. This condition is necessary and sufficient for three biflecnodes.

We have seen that any axis of g_3 determines a ruled cubic two-way T_3 whose generators are lines bisecant to R and which mark out on R pairs of an involution of quadratics apolar to a quadratic k on R. The lines of any T_3 in F_4 project from a line into the lines of a rational curve of class 3. The directrix of T_3 projects into the double line of this curve. Hence we have:

Lines joining pairs of an involution, — quadratics apolar to a quadratic k, — on $Rp\pi$ touch a rational curve ρ_k of class 3.

This theorem may easily be proved analytically in the plane. Let $Rp\pi$ be

$$\rho\, x_i = (\alpha_i\, t)^4.$$

The line joining the two points t_1, t_2 is

$$\rho\, \xi_i = |\, \alpha_k\, \alpha_l\, |\, \{ (\alpha_k\, t_1)^3 (\alpha_l\, t_2)^3 + (\alpha_k\, t_1)^2 (\alpha_k\, t_2) (\alpha_l\, t_1) (\alpha_l\, t_2)^2$$
$$+ (\alpha_k\, t_1)(\alpha_k\, t_2)^2 (\alpha_l\, t_1)^2 (\alpha_l\, t_2) + (\alpha_k\, t_2)^3 (\alpha_l\, t_1)^3 \}$$

$$(i, k, l = 1, 2, 3; \ 2, 3, 1; \ 3, 1, 2). \tag{7}$$

If t_1 and t_2 are defined by
$$(a\,t)^2 + \lambda\,(b\,t)^2 = 0, \tag{8}$$
the symmetric functions $t_1 t_2$, $t_1 + t_2$ and 1 are given as linear functions of λ by (8), and these substituted in (7) give the ξ's as cubic functions of λ.

We now find the double line of the cubic $\rho_k^{(3)}$. The directrix p_k of the T_3 of bisecant lines to R joining pairs of points apolar to k projects into this double line. Let $(m\,t)$ and $(n\,t)$, or simply m and n, be the roots of $(k\,t)^2 = 0$. Then, in accordance with the notation above,
$$m\,n \equiv (k\,t)^2 = |a\,b|\,(a\,t)(b\,t).$$
The tangents to R at m and n meet p_k. It is easy to verify that the points of p_k define on R quartics of the form
$$(l\,t)^4 \equiv m\,n\,(\mu_1\,m^2 + \mu_2\,n^2) = 0. \tag{9}$$
The double line of $\rho_k^{(3)}$ is then
$$\rho\,x_i = |\alpha_i l|^4, \tag{10}$$
where μ_1/μ_2 is the variable parameter. The quartic
$$(l\,t)^4 = m\,n\,(\mu_1\,m^2 + \mu_2\,n^2)$$
has the neutral pair
$$\mu_1\,m^2 - \mu_2\,n^2 = 0. \tag{11}$$

The form (9) may be determined directly from the pencil
$$(a\,t)^2 + \lambda\,(b\,t)^2 = 0.$$
There are two values of λ for which this is a perfect square; these are given by
$$(a_0 + \lambda\,b_0)(a_2 + \lambda\,b_2) - (a_1 + \lambda\,b_1)^2 = 0.$$
Let the roots of this be λ_1 and λ_2. Then
$$(a\,t)^2 + \lambda_1\,(b\,t)^2 \equiv m^2,$$
$$(a\,t)^2 + \lambda_2\,(b\,t)^2 \equiv n^2.$$
(11) must be of the form $(a\,t)^2 + \lambda\,(b\,t)^2$. We write then
$$(a\,t)^2 + \lambda\,(b\,t)^2 \equiv \mu_1\,m^2 - \mu_2\,n^2 = \mu_1\,[(a\,t)^2 + \lambda_1\,(b\,t)^2] - \mu_2\,[(a\,t)^2 + \lambda_2\,(b\,t)^2].$$
Hence:
$$\mu_1 - \mu_2 = 1, \qquad \mu_1\lambda_1 - \mu_2\lambda_2 = \lambda,$$
and we can take, to within a factor,
$$\mu_1 = \lambda - \lambda_2, \qquad \mu_2 = \lambda - \lambda_1,$$
and (9) becomes
$$(l\,t)^4 = |a\,b|\,(a\,t)(b\,t)\,.\,[\,|a\,b|^2\,.\,(a\,t)^2 - |a\,a'|^2\,.\,(b\,t)^2$$
$$+ \lambda\,\{\,|b\,b'|^2\,.\,(a\,t)^2 - |a\,b|^2\,.\,(b\,t)^2\}\,]. \tag{12}$$
The λ in (12) is the same as the λ in (7) and (8).

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Through any point, a_1, of g_3 we have a unique bisecant line to R, and two axes of g_3. There are thus determined, as we have seen, three mutually apolar quadratics, q_1, q_2, q_3, on R. But we have the additional theorem:

The quartic defined on R by the point a_1 of g_3 is the product $q_2 q_3$. Its neutral pair is q_1.

§ 13. *The Nodes of $Rp\pi$.*[*]

A line p meets g_3 in three points, thus determining three lines bisecant to R and meeting p. Projecting from p, the three quadratics so determined are the nodes of $Rp\pi$. Associated with every node of $Rp\pi$ is a bisecant line to R, and a catalectic set of the fundamental involution of $Rp\pi$ determined by the point in which this bisecant to R meets π.[†]

To use the notation of the previous section, let us call the point in which a bisecant line to R meets p $a_1^{(1)}$. This bisecant meets R in two points defined by a quadratic $q_1^{(1)}$. There are two axes of g_3 through $a_1^{(1)}$; let us call these $p_2^{(1)}$ and $p_3^{(1)}$. We have then the configuration of § 12, which is determined uniquely by any point $a_1^{(1)}$ of g_3: three lines of g_3 on a syzygetic plane in F_4, the three vertices $a_1^{(1)}$, $a_2^{(1)}$, $a_3^{(1)}$ defining by their bisecants to R three mutually apolar quadratics. Let us consider for a moment the line $p_3^{(1)}$ through $a_1^{(1)}$. Bisecant lines to R which meet $p_3^{(1)}$ lie on a cubic two-way T_3 of which $p_3^{(1)}$ is the directrix. Spaces on p and on $p_3^{(1)}$ cut out of this T_3 an involution of generators;[‡] the spaces on the double lines of this involution meet R in two points, each counted twice; that is, they are spaces on p bitangent to R. Hence $p_3^{(1)}$ is a line on two bitangent spaces to R from p; similarly $p_2^{(1)}$ is a line on the other two bitangent spaces to R from p. Again, the set of the fundamental involution of $Rp\pi$ defined by the point $a_1^{(1)}$ of p is given on R by the two pairs of tangent lines meeting $p_2^{(1)}$ and $p_3^{(1)}$ (§ 12). We have, therefore, from the theory of § 12:

Associated with a node of $Rp\pi$ is a definite catalectic set of the fundamental involution. The node determines a definite pairing off of the double tangents 1, 2, 3, 4 of $Rp\pi$, say 12, 34. The catalectic set of the fundamental involution associated with the node is given by the tangents to $Rp\pi$ (not double tangents) from 12 and 34. The quadratic giving the node is the neutral pair of this set.

* Cf. Marletta, *Annali di Mat.*, *loc. cit.*, p. 111.
† Stahl, *Math. Annalen*, 38, p. 571.
‡ Veronese, *Math. Annalen*, 19, p. 231.

Again:

Lines on a point 12 *cut out from* $R\,p\,\pi$ *two quadratics, each of which is apolar to the points defined by the pair of tangents from* 12. *The line* 12, 34 *cuts out of* $R\,p\,\pi$ *a quartic of a syzygetic pencil, of which the sextic covariant is the product of the nodal parameters and the associated set of the fundamental involution. Lines joining pairs of points whose parameters are apolar to the parameters of the node touch a rational curve of class* 3 *having* 12/34 *for double line. Two points of the diagonal triangle of sets of the syzygetic pencil of quartics on* $R\,p\,\pi$ *are at* 12 *and* 34; *that is, this syzygetic pencil is cut out of* R *by pairs of lines on either* 12 *or* 34. *The correspondence thus determined in either of these pencils of lines is* (2, 2).

§ 14. *The Fundamental Involution of* $R\,p\,\pi$ *in Lines, or of* $\Sigma\,\pi$ *in Points.*

Points of π determine on $R\,p\,\pi$, by means of tangents on them, an involution $I_{2,4}$ of binary sextics. The involution $I_{3,3}$ of all sextics apolar to sets of this $I_{2,4}$ is easily determinable algebraically when the fundamental involution of $R\,p\,\pi$ is given.

Let us consider a rational quartic $R\,a\,\alpha$ in space. The surface $\gamma_3\,a\,\alpha$ is the locus of pure second osculants of $R\,a\,\alpha$, or is the locus of planes in α cutting out harmonic pairs from $R\,a\,\alpha$. The four stationary points of $R\,a\,\alpha$ are the projections from a of the four points

$$(a\,t)^4 = 0$$

defined on R by a. All plane sections of $R\,a\,\alpha$ are apolar to $(a\,t)^4$. The sextic covariant T of $(a\,t)^4$ is the three pairs of points on $R\,a\,\alpha$ such that the osculating plane at each point of the pair passes through the other. The chords of $R\,a\,\alpha$ through these pairs of points meet in a point, the point $s\,.\,a\,\alpha$, the trace on α of the unique s-line on a. $s\,.\,a\,\alpha$ is the triple point of $\gamma_3\,a\,\alpha$. The three chords are the double lines of $\gamma_3\,a\,\alpha$.[*]

$S\,a\,\alpha$ is the developable of tangents to $R\,a\,\alpha$. It is of order 6. Any linear complex in α contains six lines of $S\,a\,\alpha$, thus determining a binary sextic on $R\,a\,\alpha$, the points of contact of the six lines of $S\,a\,\alpha$. A linear complex being in general determined by five lines, it follows that the six points of contact of lines on $S\,a\,\alpha$ that lie in a linear complex are in an involution $I_{5,1}$; that is, they are apolar to a definite sextic on $R\,a\,\alpha$.[†] This sextic must be a covariant of $(a\,t)^4$, and can be nothing but the covariant T. In particular, we have:

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[*] Marletta, *Annali di Mat.*, lcc. cit., p. 102.   [†] Study, *Leipz. Berichte*, 1886, p. 7.

Lines of $R\,a\,\alpha$ which meet a given line in α determine on $R\,a\,\alpha$ six parameters apolar to T.

Let us project from any point b of α on a plane π on α, thus obtaining a rational plane quartic $R\,a\,b\,\pi$. We have from the above theorem:

Lines of $R\,a\,b\,\pi$ on a point of π determine on $R\,a\,b\,\pi$ six parameters apolar to the sextic covariant T of $(a\,t)^4$.

Now $(a\,t)^4$ is *any* set of the fundamental involution of $R\,a\,b\,\pi$. It follows that:

Lines of a rational quartic $R\,p\,\pi$ on a point of π determine on $R\,p\,\pi$ a set of six parameters apolar to all sextic covariants T of sets of the fundamental involution of $R\,p\,\pi$.

The sextic covariant of a quartic

$$a + \lambda\,b = 0$$

is of the form

$$T(a) + \lambda\,M(a\,b) + \lambda^2 M(b\,a) + \lambda^3\,T(b) = 0, \qquad (1)$$

where we indicate by $T(a)$, $T(b)$ the sextic covariants of a and b respectively and M is a symbol for an intermediate covariant of a and b determined from T.

The four forms which are the coefficients of (1) *generate the fundamental involution $I_{3,3}$ of $R\,p\,\pi$ taken in lines, since these four forms must be linearly independent.*

§ 15.   *The Curves $S\,a\,p\,\pi$ and $\Sigma\,\pi'\,p\,\pi$.*

The projections from a line p of space sections of S include first osculants of $R\,p\,\pi$ as special cases, as we have seen (§ 3); similarly, the projections from a line p of plane sections of Σ include second osculants of $R\,p\,\pi$ as special cases. The space of R at any point meets S in the first osculant of that point, and the plane of R at any point meets Σ in the pure second osculant of that point.

The following theorems as to curves traced on S are sufficiently obvious:

Any curve, C, on S meeting the lines of S once is perspective to S and is rational. $C\,p\,\pi$ is perspective to $R\,p\,\pi$ (taken in lines).

Any curve, K, on S touches all the planes of R and touches each plane of R as often as it meets a line of S.

For a plane of R contains two consecutive lines of S.

Let K meet each line of S k times. Then we have:

The curve $K\,p\,\pi$ has the stationary lines of $R\,p\,\pi$ as k-fold tangents.

For instance, a general quadric of F_4 meets S in a curve K of order 12; K meets each line of S twice. The curve $K\,p\,\pi$ is a 12-ic curve on π having

the stationary lines of $Rp\pi$ for double tangents and having eight cusps at the projections of the points in which the given quadric spread meets R.

Any curve drawn on S osculates the spaces of R as often as it meets the lines of S.

For a space of R contains three consecutive tangents of R.

A space α meets S in a rational curve of class 4 and order 6. $S\alpha$ meets each line of S once. Hence:

The curves $S\alpha p\pi$ are perspective to $Rp\pi$ taken in lines. They have cusps at the projections of the four points where α meets R. They are rational curves of class 6 and order 6; they have four cusps and four stationary lines, six nodes and six double tangents. They touch the six stationary lines of R. A curve $S\alpha p\pi$ is uniquely determined on π when $Rp\pi$ and the quartic

$$(\alpha\,t)^4 = 0$$

defining its cusps, are given.

If $Rp\pi$ is

$$\rho\,x_i = (\alpha_i\,t)^4$$

and the quartic giving the cusps of $S\alpha p\pi$ is·

$$(\alpha\,t)^4 = 0,$$

then $S\alpha p\pi$ is

$$\rho\,x_i = |\,\alpha_i\,\alpha\,|\,(\alpha_i\,t)^3\,(\alpha\,t)^{\varepsilon}.^{*}$$

Let the space α meet p in the point a. Then $(a\,t)^4$ is the unique set of the fundamental involution of $Rp\pi$ which is apolar to $(\alpha\,t)^4$. $Ra\beta$ is a rational quartic in the space β. Let π be chosen on β and let p meet β in b. Then $Ra\beta p\pi$ is $Rp\pi$. $Sa\alpha\beta b\pi$ is $S\alpha p\pi$. $Sa\alpha\beta$ is the section by the plane $\alpha\beta$ of the developable, $Sa\beta$, of $Ra\beta$. The points $(a\,t)^4 = 0$ are the stationary points on $Ra\beta$, and the tangents at these points are inflectional generators on the developable of $Ra\beta$. We have, then:

The quartic $(a\,t)^4$ on $Rp\pi$ determines a unique set, $(a\,t)^4 = 0$, of the fundamental involution of $Rp\pi$ to which it is apolar. The tangents to $Rp\pi$ at the points $(a\,t)^4 = 0$ pass through the four points of inflection of $S\alpha p\pi$.

We omit for the sake of brevity the consideration of the specializations of the curve $S\alpha p\pi$ when the quartic α is specially chosen. Reductions in the order of $S\alpha p\pi$ occur for coincidences among the roots of α; if all four roots

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* § 3.

coincide at $t = \tau$, we have the first osculant of $R\,p\,\pi$ at the point τ. The above theorems may be stated for first osculants; they are known, but they serve to show the close analogy between first osculants and the curves with which we are dealing.

We may state the further theorems, omitting the proof, which is easy:

The six nodes of $S\,\alpha\,p\,\pi$ are on a conic.

The six double tangents of $S\,\alpha\,p\,\pi$ touch a conic.

Let π', a plane in F_4, be the axis of the pencil of spaces

$$(\alpha\,x) + \lambda\,(\beta\,x) = 0.$$

Then $\Sigma\,\pi'\,p\,\pi$ is a rational line-quartic on π associated with the pencil of binary quartics on $R\,\pi\,p$:

$$(\alpha\,t)^4 + \lambda\,(\beta\,t)^4 = 0.$$

The curves $S\,[\alpha + \lambda\,\beta]\,p\,\pi$ pass through the six cusps of $\Sigma\,\pi'\,p\,\pi$. $\Sigma\,\pi'\,p\,\pi$ is given in points (§ 3) by

$$\rho\,X_i = |\,\alpha_i\,\alpha\,|\ |\,\alpha\,\beta\,|\ |\,\beta\,\alpha_i\,|\ (\alpha\,t)^2\,(\beta\,t)^2\,(\alpha_i\,t)^2.$$

On the Analytical Basis of Non-Euclidian Geometry.

By W. H. Young, Sc. D., F. R. S.

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§ 1. *Introductory.*

There is, it seems to me, a tendency at the present time to throw dust in the eyes of the mathematical public or rather of the schoolboy public, in respect of the step taken which corresponds to the assumption of Euclid's XIth Axiom. To efforts, now recognized by all mathematicians to be abortive, towards proving the XIth Axiom, have succeeded treatments of the subject matter of Euclid for future engineers and others which gloze over the difficulty this axiom involves. I have recently proposed[*] to emphasize the empirical character of the axiom by giving to it a form which challenges attention, a form, moreover, having the double advantage that it relates to bounded space, and can be experimentally verified by the dullest schoolboy, even by one to whom the idea of an angle represents an incompletely solved difficulty.

If on the other hand we leave the region of school-books, we find for the most part an air of unreality and half-heartedness in the treatment of the subject. Almost all writers tacitly or expressly assume a knowledge of Euclidian geometry, and express themselves as if the Euclidian hypothesis were the more natural. But further, the possibility of a non-Euclidian system is regarded as demonstrated by means of a correspondence between a selected set of Euclidian entities with the elements of non-Euclidian space. Other writers have made the assumption that Euclidian geometry holds in the smallest parts (im Kleinen).[†]

The object of the present paper is to show how the refinements of modern analysis enable us to give a perfectly rigid and consistent account of the three geometries which arise when, retaining in all cases Euclid's remaining axioms, we adopt one of the three alternatives which may take the place of Euclid's Fifth Postulate, sometimes called the XIth Axiom. The retained axioms may

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[*] "On a Form of the Parallel Axiom," 1910, *Quarterly Journal*, Vol. XLI, pp. 353–363.

[†] Such a method is like that of teaching a foreign language through your own. Non-Euclidian geometry from such a standpoint is relegated to the place of a dead language.

be expressed in different forms. Adopting Hilbert's classification, they consist of four groups:

      I) The Axioms of Classification;
      II) The Axioms of Order;
      III) The Axioms of Congruence;
      IV) The Axioms of Continuity.

Hilbert's fourth group consists precisely of the Axiom of Parallels; that is, the equivalent of Euclid's Fifth Postulate.* The form of the three alternative axioms, characterizing the three geometries under discussion, referred to above in the first paragraph, is as follows:

Let it be assumed that there is one triangle in the bounded convex space we are considering, such that the length of the line joining the middle points of the two sides is

    1) *less than (Lobatchefsky-Bolai geometry),*
    2) *equal to (Euclidian geometry),*
    3) *greater than (Riemann geometry)*

half the third side.† It then follows that, in the respective cases, the sum of the angles of the triangle is less than, equal to, or greater than two right angles;‡ and conversely, if this holds, the corresponding statement in the axiom is true. Also the same is true of every triangle.§

Moreover, the defect of a variable triangle QAP, that is the difference between the sum of its angles and two right angles, has zero as limit, if the point P moves along a straight line towards A as limiting point.‖ Hence, it easily follows that, *if we have a triangle of varying shape and position, which is in course of shrinking up to a point, its defect has zero as limit,* since ¶ the defect of a triangle is less than that of another triangle containing it.

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* D. Hilbert: "Grundlagen der Geometrie," Leipzig, Teubner, 2nd Edition, 1903. I should like here to refer to Enriques' collection of essays by various Italian authors, "Quistioni Riguardanti la Geometria Elementare," Bologna, 1900; the second volume of the German translation is now in the press. There are various accounts of such sets of axioms in English; *e. g.,* in Halsted's "Rational Geometry." See also A. N. Whitehead's two contributions to the Cambridge Tracts in Mathematics.

† *Loc. cit.,* § 2.   ‡ *Loc. cit.,* § 15.   § *Loc. cit.,* § 16.   ‖ *Loc. cit.,* §§ 13, 14.

¶ The classical method of proving this statement is to deduce it in steps, at each step comparing two triangles with two common vertices, say B and C, and the remaining vertices in a straight line with C, say ADC. If then, in the Lobatchefsky geometry,

$$\pi - ABC - BAC - CAB < e,$$

we have at once,

$$(\pi - ABD - ADB - BAD) + (\pi - DBC - DCB - BDC) < e.$$

Since each of the expressions in brackets is positive, each must be $< e$. Similarly, the statement may be proved in the other geometries.

The line of argument which I have adopted is one of those indicated by Killing in a couple of pages of his "Grundlagen der Geometrie." The best justification for the detailed account here given will be found in the comparison, made by the reader himself, between the paper before him and these pages of the "Grundlagen der Geometrie." One other writer appears to have treated the subject from a similar point of view, Girard in his Thèse. The method he employs is, however, quite different, and, as it appears to me, less natural from the point of view of analysis. The definition of the sine, cosine and tangent as limits presents itself inevitably to the modern analyst, whereas this can hardly be said of the definitions given by Girard.

On the basis here given the whole edifice of non-Euclidian or Euclidian geometry can be securely built. The discussion terminates as soon as we have obtained the formulæ connecting the sides and angles of a triangle. For the convenience of readers the inequalities are stated for the Lobatchefsky case. The discussion here given is, however, perfectly general. It is, moreover, easy to deduce all the results in three dimensions from the plane formulæ here obtained, ranging from the theorem, sometimes tacitly assumed without justification, that there is in all three geometries such a thing as the angle between two planes, to all the interesting properties in Clifford's Theory of Parallels in Elliptic Space, and of the Grenzfläche in the Lobatchefsky-Bolai geometry.

§ 2. *The Tri-Rectangle.*
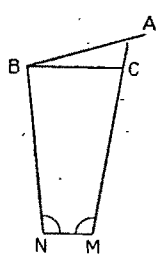
FIG. 1.

In the Lobatchefsky geometry the sum of the angles of any triangle is less than two right angles, and therefore that of the angles of a quadrilateral is less than four right angles. Hence, if the quadrilateral has a pair of opposite sides *BN*, *CM* (Fig. 1) equal, and the angles at *N* and *M* right angles, the other angles, being by symmetry equal, are both acute. It follows that the perpendicular from *B* to *BN* falls outside the quadrilateral. Hence, if from any point *A* on one of two straight lines *AM*, *BN*, perpendicular at *N* and *M* to the same straight line, we drop a perpendicular *AB* on to the other line *BN* to form a quadrilateral *ABNM* with three right angles, the side *BN* is less than *AM*. That is, *in a quadrilateral with three right angles, a side between two right angles is less than the opposite side.*

§§ 3–9. *The Sine-Ratio and the Sine.*
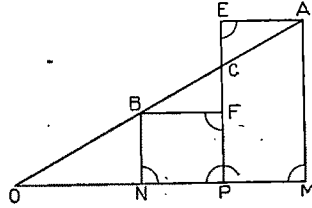
FIG. 2.

§ 3. Take any acute angle, let O be its vertex, and from A and B, any two points on one of the arms, draw AM, BN, perpendicular to the other arm. Let C be the middle point of AB, so that

$$2\,OC = OA + OB.$$

Draw CP perpendicular to the other arm, and AE perpendicular to CP. On the other side of C from E on the straight line CP take F, so that

$$CF = CE.$$

Then the triangles CFB, CEA are congruent, since they have equal angles at C and the containing pair of sides of the one equal to that of the other. Hence, the angle at F, like that at E, is a right angle.

From the result at the end of the preceding article it follows that

$$EP < AM,$$
$$FP < BN,$$

and therefore, since the sum of FP and EP is twice CP,

$$2\,CP < AM + BN.$$
$$\therefore \quad \frac{CP}{OC} < \frac{AM + BN}{OA + OB}.$$

Now let us denote the ratio of AM to AO by a, that of BN to OB by b, and that of CP to OC by c. Then the last inequality becomes

$$c < \frac{a \cdot OA + b \cdot OB}{OA + OB} \qquad (1),$$

whence, if we know from other sources that

$$b < c,$$

it follows that

$$c < a.$$

§ 4. Now let us reflect Fig. 2 in the line OM, and denote the reflections by the same letters as before, distinguishing by a dash. Then, if B is the middle point of OC, B' is the middle point of OC', so that (§ 1)

$$BB' < \tfrac{1}{2} CC',$$

whence, since the line at which we reflect bisects both BB' and CC',

$$BN < \tfrac{1}{2} CP.$$

Hence, since

$$OB = \tfrac{1}{2} OC,$$
$$b < c.$$

§ 5. From the last two articles it follows that, if

$$OB : OC : OA = 1 : 2 : 3,$$
$$b < c < a.$$

Hence, it follows by induction that, *if OB, OC and OA are in the ratio of any three consecutive integers, b, c and a are in ascending order of magnitude.* For, suppose this to have been proved for the three consecutive integers $(n-1)$, n, $(n+1)$, and let

$$OB : OC : OA = n : n+1 : n+2.$$

Then we know that

$$b < c;$$

therefore, by § 3,

$$c < a,$$

which proves the statement for the next trio of consecutive integers. But it has been proved for the three successive integers 1, 2, 3; hence the induction is complete, and it is true for any three consecutive integers.

§ 6. Hence, if OA be divided into any number of equal parts, the ratio of the perpendicular from any point on OA to its distance from O, or as we shall say, *the sine-ratio*, increases monotonely from point to point of division passing from O to A. Since this is true at each stage when we divide successively into 2, 4, 8, ... parts, it follows that, if x is the sine-ratio at any point of binary division X, and y the sine-ratio for another of these points Y, then if X lies between O and Y,

$$x < y.$$

§ 7. The binary points of division, though dense everywhere,[*] are only

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[*] For the explanation of this and other terms borrowed from the Mengenlehre, the reader may refer to "The Theory of Sets of Points," by W. H. Young and Grace Chisholm Young, Cambridge University Press (1906).

countable. In order to prove further that the statement at the end of the preceding article is true when X and Y are not neccessarily points of binary division, we have to show that x is a continuous function of the position of the point X.

Now since the angles of a triangle are together less than two right angles, the angle OAM is acute. Also, since (§ 1) the defect from two right angles of the sum of the angles of a triangle has zero for limit when one of the vertices moves along a straight line up to the other as limiting point, it follows that the sum of the angles of the quadrilateral $AMPE$ of Fig. 2 is as near as we please to four right angles, provided A is near enough to C. Hence, we may assume that A is sufficiently near C for the angle at A, the only angle of the quadrilateral which is not a right angle, to be greater than the acute angle OAM.

This being so, E falls on the other side of C from P, and therefore

$$CP < PE < AM,$$

using the result of § 2. But

$$AM < AC + CP + PM;$$

thus

$$CP < AM < CP + AC + PM.$$

Now when A moves up to C as limit, M moves up to P, both moving on straight lines, viz., the arms of the angle. Hence, AC and PM each have the unique limit zero. Thus the extremes in the last inequality have the same limit CP, so that AM has also CP as unique limit.

Since also OA has OC as unique limit, the ratio AM/OA has as limit the ratio CP/OC.

This proves that the *sine-ratio is a continuous function.*

§ 8. Now *a continuous function which is monotone with respect to an everywhere dense set of points is a monotone function.* For let X lie between O and Y,
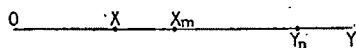
FIG. 3.

and let Y_n be any point of the everywhere dense set between X and Y, and X_m any point of the same set between X and Y_n. Then denoting the values of the function at these points by x, x_m, y_n and y, and supposing the function to *decrease* as we move from one point to another of the everywhere dense set in the direction towards O,

$$x_m < y_n.$$

Let X_m move along a sequence of the points of the everywhere dense set in the direction of O and have X as limiting point. Then the quantities x_m diminish down to their unique limit, which is x, since the function is continuous. Hence,

$$x < x_m.$$

Similarly, if Y_n moves towards Y as limit along points of the same set,

$$y_n < y.$$

Hence,

$$x < x_m < y_n < y.$$

This shows that the function is a monotone function.

§ 9. We have now shown that the sine-ratio is a continuous function which is monotone decreasing as the point X approaches O. Hence, this ratio has a unique limit at the point O, and this limit is less than any value of the ratio. This limit we shall call *the sine of the angle AOM*, and write in the usual way sin AOM.

§§ 10–12. *The Cosine-Ratio and the Cosine.*

§ 10. Let AM and BN be any two perpendiculars from one of the arms ABO of an acute angle at O on to the other arm, B being nearer to O than A. Bisect NM at P, and draw PC perpendicular to NM, meeting OA (as of course it must do, since it can not meet AM nor BN) at C.
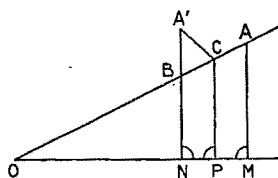
FIG. 4.

Then, since the sum of the angles of a quadrilateral is less than four right angles, the angle OBN is greater than the angle OAM. Hence, if A' is the reflection of A in CP, the angle $BA'C$, being equal to OAM, is less than the angle $A'BC$, that is OBN. Hence,

$$BC < CA',$$

that is,

$$BC < CA.$$

§ 11. If the points B and N coincide with the point O itself, the argument of the preceding article still holds. Hence, if we divide OM into any number

of equal parts and construct the perpendiculars to OM at the points of division, these divide OA in such a way that the stretch between two consecutive points of division increases monotonely as we recede from O.

Denoting the points of division on OM by P_1, P_2, ..., P_{n-1}, and those on OA by the letter Q with the same indices, we have

$$r \cdot OQ_{r-1} = (r-1) \, OQ_{r-1} + OQ_{r-1} = (r-1) \, OQ_r - (r-1) \, Q_r Q_{r-1} + OQ_{r-1}.$$

But

$$OQ_{r-1} < (r-1) \, Q_r \, Q_{r-1},$$

since it consists of $(r-1)$ parts each less than $Q_r \, Q_{r-1}$. Hence,

$$r \cdot OQ_{r-1} < (r-1) \, OQ_r,$$

so that

$$OQ_{r-1}/(r-1) < OQ_r/r,$$

whence

$$OP_{r-1}/OQ_{r-1} > OP_r/OQ_r.$$

Calling the ratio here considered — viz., that of the base OP_r to the hypotenuse OQ_r of the right-angled triangle $OP_r Q_r$ — *the cosine-ratio* at the point P_r, this shows that, if OA be divided into any number of equal parts, *the cosine-ratio is a function which, as we move along from point to point of division towards the vertex O, increases monotonely.*

§ 12. Hence, if we divide OA by continued bisection, and so get the set of points of binary division, the cosine-ratio at the points of binary division is a function which decreases monotonely as we approach O.

Hence, by § 8, if this ratio is a continuous function of the point, it is a monotone increasing function as the point approaches the vertex O. Now it is evident that the cosine-ratio is continuous, for if C be any point between B and A, and P be the foot of the perpendicular from C on OM (Fig. 4), this perpendicular does not meet AM nor BN, since two angles of a triangle are together less than two right angles; therefore it meets the stretch NM, so that P lies between M and N. Hence, if M moves up to N as limiting point, and therefore certainly passes P, A must pass C. Thus A moves up to B as limiting point, since it can not have as limiting point any point C on the other side of B from O. Thus not only OM has ON for limit, but OA has OB, and therefore OM/OA has ON/OB for limit. *This proves the continuity, and therefore also the monotony of the function constituted by the cosine-ratio OM/OA, as M moves towards O.*

§ 12. Since the cosine-ratio increases monotonely as the point M approaches O, it has a unique limit at the point O, which is at the same time its upper bound. This limit we shall call *the cosine of the angle AOM*, and write in the usual way $\cos AOM$.

§ 13. *The Tangent, Cotangent, Secant and Cosecant.*

We have seen that the sine-ratio decreases and the cosine-ratio increases; hence it follows that their ratio, which is the limiting value of the ratio of the perpendicular AM to the base OM, decreases, and has therefore a unique limit at O, which is at the same time its lower bound. This limit is called *the tangent of the angle AOM*, and is denoted by $\tan AOM$.

The reciprocals of the sine, cosine and tangent are called *the cosecant, secant and cotangent*, or $\operatorname{cosec} AOM$, $\sec AOM$, $\tan AOM$, respectively.

§ 14. *Bounds of the Sine and Cosine.*

Since the angles of a triangle are less than two right angles, so that in a right-angled triangle the right angle is the greatest angle, and therefore the hypotenuse the greatest side, it follows that the ratios whose limits are respectively the sine and cosine are both less than unity, and therefore *both the sine and the cosine of an acute angle lie between 0 and 1, both inclusive.* Again, since the base is less than the hypotenuse, the ratio of the perpendicular to the hypotenuse is less than the ratio of the perpendicular to the base. Hence, *the sine is less than or equal to the tangent.*

§ 15. *Monotony of the Sine, Cosine and Tangent.*
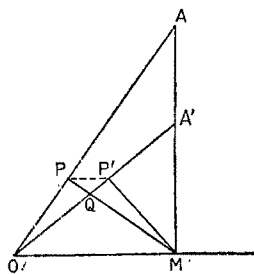
Fig. 5.

Let AOM be any acute angle, and AM perpendicular to OM. Then, if $A'OM$ is any smaller angle, the ray OA' falls inside the angle AOM, and therefore cuts AM, say at A'. Therefore,

$$AM/OM > A'M/OM.$$

Since this is true for every position of A as it moves up to O as limiting point, we get, proceeding to the limit,

$$\tan AOM \geq \tan A'OM.$$

Thus *the tangent increases as the acute angle increases.*

Again, since the angle at M is a right angle, the angle $AA'O$ is obtuse. Hence,

$$OA > OA',$$

so that

$$OM/OA < OM/OA',$$

whence, as before, proceeding to the limit,

$$\cos AOM \leq \cos A'OM.$$

Thus *the cosine decreases as the angle increases.*

Again, if MP be perpendicular to OA, P falls between O and A, since otherwise one of the angles made by MP with OA would be less than AOM or than OAM (both of which are acute angles), since the exterior angle of a triangle is greater than the interior opposite angle.

Since P lies between O and A, MP meets OA' inside the triangle OAM, say at Q. Then $MP > MQ >$ the perpendicular MP' from M on OA'. Hence,

$$MP/OM > MP'/OM.$$

Since this is true for each position of M, as M moves up to O as limiting point, we get, proceeding to the limit,

$$\sin AOM \geq \sin A'OM.$$

This proves that *the sine of an acute angle increases as the angle increases.*

Thus, as the angle increases, the sine, cosine and tangent are monotone functions of the angle, the sine and tangent increasing and the cosine decreasing. (N. B. It will be shown later that the alternative sign of equality in the inequalities proving the monotony of the sine, cosine and tangent is inadmissible.)

§ 16. *Values of the Sine, Cosine and Tangent of any Acute Angle.*

It follows from the definitions of the sine, cosine and tangent, as the unique limits of certain positive ratios, that they are never negative. It will now be shown that, the angle being as hitherto an acute one, *the sine, cosine and tangent are themselves positive, and that the sine and tangent have zero as unique limit, and the cosine has unity as unique limit, when the angle decreases indefinitely.*

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Divide a right angle by continued bisection into n equal parts, and divide an adjacent right angle in the same way (Fig. 6). Let $OK_0 = OK_{2n}$ be equal lengths on each side of the common arm OK_n of the two right angles, and take OK_2, OK_4, ... on the alternate arms of the parts each equal to OK_0. Then, if we join $K_0 K_2$, $K_2 K_4$, ... all round to K_{2n}, we get a polygon whose last side is $K_{2n} K_0$, and whose n other sides are each equal to $K_0 K_2$. Hence,

$$n \cdot K_0 K_2 > K_{2n} K_0 > 2 OK_0.$$

But, if $K_0 K_2$ meet the intermediate dividing line in K_1, so that the angle $K_0 OK_1$ is the first of the n equal parts into which we had divided the first right angle, the angles $OK_1 K_0$ and its adjacent angle $OK_1 K_2$ are equal, and each is a right
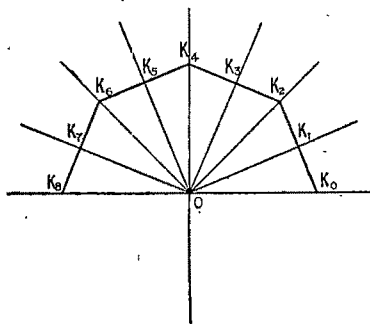
FIG. 6.

angle. Also $K_0 K_1 = K_1 K_2$, all this following from the congruence of the triangles $OK_1 K_0$ and $OK_1 K_2$. Hence,

$$2n \cdot K_0 K_1 > 2 OK_0,$$

so that

$$K_0 K_1 / OK_0 > 1/n.$$

Since this is true for every position of the point K_0 as it moves up to O as limiting point, we get, proceeding to the limit,

$$\sin K_0 OK_1 \geq 1/n.$$

Now whatever acute angle be given, we can find n so large that the given angle is greater than the angle $K_0 OK_1$, which is the n-th part of a right angle. Since the sine increases with the angle, it follows that the sine of the given angle is $\geq 1/n$, and is therefore positive.

Since the tangent of an acute angle is greater than the sine, this proves that both the sine and the tangent of an acute angle or positive, and therefore that the cosine, which is the quotient of the sine by the tangent, (since neither is zero) is also positive.

§§ 17, 18. *Limiting Values of the Sine, Cosine and Tangent.*

§ 17. To find the unique limits approached by the sine, cosine and tangent when the angle diminishes without limit, let us return to Fig. 5. Let OM be any stretch and AM a line perpendicular to OM and of one n-th of the length of OM. Then

$$0 < \sin AOM < \tan AOM < AM/OM < 1/n,$$

and, since $OA < OM + MA$,

$$1 < OM/OA + MA/OA < \cos AOM + 1/n.$$
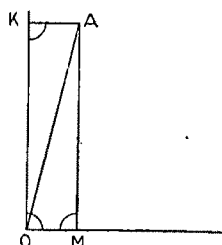
FIG. 7.

Increasing n without limit, so that the angle AOM diminishes without limit, these inequalities show that *the sine and tangent have, when the angle decreases indefinitely, the limit zero and the cosine the limit unity.* Since the sine, cosine and tangent are monotone, these limits are, of course, unique.

§ 18. We can now find the unique limits of the sine, cosine and tangent when the acute angle increases with a right angle as limit. For, let AOM be an acute angle (Fig. 7), AM perpendicular to OM, and let MOK be a right angle, and AK perpendicular to OK. Then (§ 2)

$$OK < AM, \text{ and } OM < AK.$$

Hence,

$$OK/OA < AM/OA \text{ and } OM/OA < AK/OA.$$

Letting the point A move along the arm AO of the acute angle up to O as limiting point, we get, proceeding to the limit, [*]

$$\cos AOK \le \sin AOM \text{ and } \cos AOM \le \sin AOK.$$

Using the results of the preceding article, we see that, *as the angle AOM increases up to a right angle, its cosine has the limit unity and its sine the limit zero; hence, also, its tangent has the limit $+ \infty$.*

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[*] See § 22, where it is shown that the sign of equality must be taken.

§ 19. *Sine, Cosine and Tangent of a Zero Angle and of a Right Angle.*

So far we have only defined the sine, cosine and tangent of an acute angle. We now extend our definition by assigning to these functions of the angle as values when the angle is zero or a right angle the values of their unique limits. Thus

$$\text{sine } 0 = 0, \quad \text{cosine } 0 = 1, \quad \text{tangent } 0 = 0,$$
$$\textit{sine of a right angle} = 1, \quad \textit{cosine of a right angle} = 0,$$
$$\textit{tangent of a right angle} = + \infty.$$

With these definitions the sine, cosine and tangent are monotone from the value zero to the value a right angle, and they are all continuous for the extreme values of the angle. For the value a right angle, the tangent is only continuous in the extended sense, since it has the value $= + \infty$ there.

§ 20. *Continuity of the Cosine.*
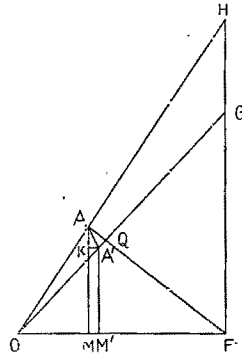
Fig. 8.

Let HOF be any acute angle and GOF any smaller acute angle inside HOF. From any point F on the common arm of the two angles draw FA
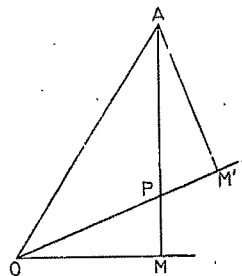
Fig. 9.

perpendicular to OH. Let FA meet OG in Q. Then, since in a right-angled triangle the hypotenuse is the greatest side, $OA < OQ$. Cut off from OQ $OA' = OA$, and draw AM and $A'M'$ perpendicular to OF.

If we rotate the ray OA' round till A' coincides with A, the angle $A'OM'$ being less than the angle AOM, OM' (Fig. 9) will cut AM, at P say. The angle APO will then be obtuse, since it is the exterior angle of a triangle PMO having a right angle at M. Hence, the point M' cannot lie on the same side of P as O, or the angles of the triangle APM' would have a sum greater than two right angles. Thus P lies between O and M'. Hence,

$$OM < OP < OM'.$$

Also (Fig. 8), since the angles OAM and $OA'M'$ are acute, and the angle OAF is a right angle and $OA'F$ greater than the obtuse angle OQF, it follows that M and M' both lie between O and F, so that the order of these points is $OMM'F$. Hence,

$$0 < \frac{OM'}{OA'} - \frac{OM}{OA} = \frac{MM'}{OA}. \tag{1}$$

Now draw $A'K$ perpendicular to AM. Then since the hypotenuse is the greatest side of a right-angled triangle,

$$A'K < A'A,$$

and since the equal angles OAA' and $OA'A$ are both acute, so that the angle $AA'Q$ is obtuse,

$$AA' < AQ.$$

Also, since the quadrilateral $A'M'MK$ has three right angles, we have, by § 2,

$$MM' < A'K.$$

Hence finally,

$$MM' < AQ.$$

Using (1), we thus get

$$0 < \frac{OM'}{OA'} - \frac{OM}{OA} < \frac{AQ}{OA}.$$

Letting the point F move along its ray OF to O as limiting point, the point A also moves up to O as limiting point, since, if L is any point on OH, the angle FLO becomes acute as soon as F has passed the foot of the perpendicular from L on OF, and therefore, when this is the case, A lies between O and L. Thus the whole figure $FQAA'KMM'$ shrinks up to the point O. Hence, we get, proceeding to the limit with the preceding inequality,

$$0 < \cos GOF - \cos HOF < \tan HOG.$$

Now when the angle HOG approaches zero, either by the ray OH turning round till it coincides with OG, or by OG turning round till it coincides with OH, tan HOG has, as has been shown in § 17, the unique limit zero. Thus cos GOF — cos HOF has the unique limit zero. In other words, *the cosine is a continuous function of the angle.*

§ 21. *Uniform Convergence of the Cosine-Ratio to the Cosine.*

Since the cosine-ratio converges monotonely to the cosine, and the cosine is a continuous function of the angle, it follows that *the cosine-ratio converges uniformly to the cosine.*[*] In other words, if AOM is a right-angled triangle and the ray OA turns round to a definite limiting position OB, while at the same time the point A moves up to O as limit, then OM/OA converges uniformly to cos BOM as unique limit.

The cosine-ratio is here considered as being a function of two variables; first the angle AOM, or say z, and secondly the length OM, or say x. Let this function be $f(x, z)$, and let the cosine be $F(z)$. Then

$$F(z) = \lim_{x=0} f(x, z),$$

by definition; but the equation

$$F(z_0) = \lim_{x=0,\ z=z_0} f(x, z)$$

expresses the uniform convergence of the cosine-ratio. In the first equation z is constant, and the limit is a single limit; in the second both x and z vary and the limit is a double limit.

Again, if AOM is a triangle of changing form and position, having a right angle at M and a hypotenuse AO which shrinks up with zero as unique limit, and if the angle AOM has a definite limit, say a, then OM/OA converges uniformly to cos a. For we only have to redraw the changing triangle AOM in a new figure in which the point O and the straight line OM are fixed, and this result follows at once from the preceding.

We shall apply this result to prove that the sign of equality must be taken in the relations found in § 18.

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* By a well-known theorem.

§ 22. *Sine and Cosine of Complementary Angles.*

In the figure of § 17, denoting the angle AOM by x and the angle KOA by y, so that $x + y$ is a right angle, the angle OAM is less than y, but approaches y as limit as A moves up to O along the fixed ray OA. Hence, by the preceding article, AM/OA converges uniformly to $\cos y$ as unique limit.

But, by definition, AM/OA converges to $\sin x$ as unique limit. Hence,

$$\sin x = \cos y.$$

In words, the sine of an angle is the cosine of its complement, and the cosine of an angle is the sine of its complement.

§ 23. *Continuity of the Sine and Uniform Convergence of the Sine-Ratio.*

Now if x changes with a unique limit x_0, its complement y changes with y_0 as unique limit, where y_0 is the complement of x_0. Hence, since $\cos y$ approaches $\cos y_0$ as unique limit, it follows from the preceding that $\sin x$ approaches $\sin x_0$ as unique limit. Thus *the sine is a continuous function of the angle.*

Hence, *since the sine-ratio converges monotonely to the sine, it converges to it uniformly.*

§ 24. *Continuity in the Extended Sense, of the Tangent, Cotangent, Secant and Cosecant, and Uniformity of Approach of the Ratios to their Limits.*

Hence, it follows that the tangent, cotangent, secant and cosecant, being the ratios of continuous non-zero functions as long as we are dealing with acute angles, are continuous functions, and therefore the convergence of the different ratios defining these functions of the angle to their limits is uniform. This is still true for the extreme values zero and a right angle, the continuity being in the extended sense, when the value $+\infty$ is allowed, and the continuity being on one side only, viz. for approach to zero from above and to a right angle from below.

§ 25, 26. *Extension to Angles of Any Magnitude.*

§ 25. With the usual convention as to signs, we now define the sine and cosine of any angle to be the unique limits approached by the sine- and cosine-ratios, viz. the ratios of the perpendicular and base respectively to the hypotenuse of the right-angled triangle AOM, got by drawing AM from a point on one arm of the angle on to the other.

That these limits are unique follows at once from the fact that in every case these ratios are numerically the same as those derived from a certain acute angle. We have in fact the usual relations,

$$\sin(-x) = -\sin x, \qquad \cos(-x) = \cos x,$$
$$\sin(180° - x) = \sin x, \qquad \cos(180° - x) = -\cos x,$$
$$\sin(180° + x) = -\sin x, \qquad \cos(180° + x) = -\cos x.$$

Also the addition or subtraction of any number of multiples of four right angles leaves the sine and cosine unaltered.

Here 180° is used for two right angles.

§ 26. With this convention as to the meaning of the terms used, it follows immediately from what has been proved that *the sine and cosine are continuous functions of the angle*, their values for any multiple of a right angle being defined as the values of their unique limits there, and that they lie between 0 and 1 inclusive. They are monotone between any two adjacent multiples of a right angle. The relation

$$\sin x = \cos y$$

holds for all values of x and y such that $x + y$ is a right angle. We have in fact the same picture as in Euclidian trigonometry.

§§ 27–37. *The Circle and Circular Measure.*
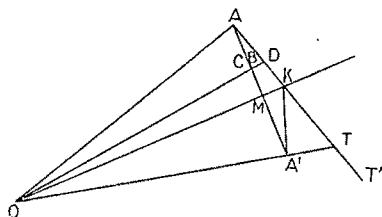
FIG. 10.

§ 27. *The Circle.* A *circle* is the locus of a point whose distance from a fixed point O, called *the center* of the circle, is constant, say r, and is called *the radius* of the circle.

Whatever bounded space we are working in, if O is an internal point of that space, there is a certain value r_0 such that for all smaller values of r, there is a point of the circle on every half-ray through O. We shall assume that r has such a sufficiently small value.

Let A be any point of the circle and AT' perpendicular to OA and of length less than one n-th of the radius r. Then if the angle AOA' is less than the angle AOT', so that OA' meets the stretch AT', say at T, $OT > OA$, so that the circle cuts the stretch OT, say at A'.

Thus every point of the straight line AT', except A, is *outside* the circle; that is, is at a distance from O *greater* than the radius r.

Let OCD be any ray inside the angle AOA', meeting AA' in C and AT in D. Then, as we saw, $OD > r$, and, since the angle $OCA > OA'A$, that is $OCA > OAA'$, it follows that $OC < OA$. Hence, the point B, in which OD meets the circle, lies between C and D.

Now the angles OAA' and $OA'A$ being equal are acute; therefore, the angle $AA'T$ is obtuse. Hence,

$$AA' < AT < AT' < r/n.$$

Also since in like manner the angles ABO, $A'BO$ are acute,

$$AB < AD \text{ and } A'B < A'D;$$

so that

$$AB + A'B < AD + A'D < AD + DT + A'T < 2AT < 2r/n.$$

§ 28. Let us now inscribe in the circle a polygon, and let p_n be its perimeter, also let the angles subtended by its sides at the center be each less than the angle AOT', above used. Also let us take another polygon, of perimeter p_{n+1}, having among its vertices all the vertices of the former polygon,

$$p_n < p_{n+1}.$$

Now each of the vertices of the latter polygon adds, as we saw, at most $2r/n$ to the perimeter; hence, if m be the number of new vertices of the second polygon,

$$p_{n+1} - p_n < 2mr/n.$$

§ 29. Thus if we inscribe a series of polygons in the circle in the manner indicated in the preceding article, by adding at each stage new vertices, the perimeters of the successive polygons will form a monotone increasing sequence of numbers, and have therefore a unique limit, viz., their upper bound. Let this be done in such a way that the integer n, used in determining the angle AOT' of § 27, increases by 1 at each stage, so that the angle subtended at the center by any side of the n-th polygon is less than the angle AOT', where $AT' < r/n$.

35

§§ 30–36.   *The Length of the Circle.*

§ 30.   Let I by the upper bound of the perimeters of all possible polygons inscribed in the circle of radius r. Then assigning any positive quantity e, we can find an inscribed polygon whose perimeter p' is greater than $I-e$. Let this polygon have m vertices; then we can choose n so large that

$$n > m\,r/e.$$

Let us form a new polygon, of perimeter p_n'', having as vertices all these of the polygon of perimeter p' and also all those of the n-th polygon described as in the preceding article of perimeter p_n.

Then p_n'' is greater than either p' or p_n. But since the number of new vertices inserted in the polygon of perimeter p_n is at most m, it follows from § 28 that $p_n'' - p_n$ is less than $2\,m\,r/n$; that is, $< 2\,e$. Thus

$$I - e < p_n'' < p_n + 2\,e.$$

Since this is true for all greater values of n, the limit of the p_n's $\geq I - 3\,e$. This being true for all values of e, the same limit $\geq I$. But the sign $>$ is inadmissible, since I is the upper bound of all possible perimeters. Thus the limit of the p_n's is I.

§ 31.   We shall next show that AT is the tangent to the circle, so that *the tangent is the line perpendicular to the radius through the point in which the latter meets the circle*, which is the point of contact of the tangent. For though the angle OAA' is acute, it has a right angle as limit, when the point A' moves along the circle up to A as limit, since the limit of the sum of the two equal angles OAA' and $OA'A$ with AOA', which decreases indefinitely, is two right angles. Hence, whatever line be drawn inside the right angle OAT'', there will be a position of A' such that AA' lies inside the angle made by that line with AT''. Thus that line lies partially inside the circle. The same is true, of course, on the other side of OA. Hence AT is the only straight line through A which lies entirely outside the circle, with the exception of the single point A, which lies on the circle. Thus AT is the tangent to the circle.

§ 32.   Hence, it easily follows that, if M be the middle point of AA', and OM meet AT in K, $A'K$ is the tangent at A', and $KA = KA'$. For the triangles AMO, $A'MO$ are congruent; since their sides are equal; therefore the point A' is the reflection of A in OK, whence the equality of the lengths and the angles in the two figures $AOMK$, $A'OMK$ follows.

§ 33. Now as A' moves along the circle up to A as limiting point, the angle OAA' has OAT' as limit, and therefore the angle KAM has zero as limit. Hence, by § 21, AM/AK converges uniformly to the cosine of zero, that is to unity, as limit. Hence,

$$AA'/(AK + KA')$$

is less than unity, but has unity as limit, when the angle AOA' is indefinitely decreased.

§ 34. Let us now circumscribe a polygon about the circle, and let p be its perimeter. Joining the points of contact, we get an inscribed polygon; let its perimeter be p'. We may suppose AA' to be one side of the latter polygon, AK and KA' the corresponding sides of the circumscribed polygon (Fig. 10). It then follows from the preceding article that p'/p is less than unity, but has unity as limit, when the angles between the radii to the points of contact of the circumscribed polygon are indefinitely diminished.

Hence the quantity I, previously defined as the upper bound of the perimeters of inscribed polygons, is the unique limit of the perimeter of a sequence of circumscribed polygons, when the angles between the radii to the consecutive points of contact diminish indefinitely, or, which is the same thing, when the points of contact form a set dense everywhere on the circle.

§ 35. It is easily seen that I *is the lower bound of the perimeters of all circumscribed polygons.* For if AK and KA' are consecutive sides of any circumscribed polygon, we obviously get a poylgon of less perimeter by inserting M, the middle point of the arc AA', as a new vertex. Thus we get a monotone decreasing sequence of perimeters, by continually bisecting the arcs between the points of contact and adding the points of bisection as new vertices. Since the limit of the perimeters is then I, it follows that the perimeter of the polygon with which we started is greater than I, so that I is the lower bound of the perimeters of all circumscribed polygons.

§ 36. This being proved, the quantity I is defined to be the *length of the circle.* Now if the angle AOK is fixed, and we diminish the radius of the circle indefinitely, the ratio AK/OA diminishes, and has as limit $\tan AOK$. Hence if p is the perimeter of the circumscribed polygon whose points of contact lie on fixed rays through the common center of the decreasing circles, p/r diminishes. Since this is true for all sets of fixed rays, it is true when we replace p by its lower bound. Thus I/r diminishes, and has therefore a unique

limit when r is indefinitely diminished; viz., its lower bound. *This limit we denote by* 2π.

§ 37. If the angle AOK is one n-th of two right angles, it follows from the preceding article that, denoting this angle by a_n, $2n \tan a_n$ diminishes monotonely as n increases and has 2π as limit.

Similarly if, instead of taking a complete inscribed polygon, we take that part of it which is bounded by radii enclosing any fixed angle, we see that the upper bound of the length of the arc of an inscribed polygon and the lower bound of the length of the arc of a circumscribed polygon are equal, and that the common value is the unique limit of the length of the arc either of an inscribed or of a circumscribed polygon, when the points of contact of the latter or the vertices of the former become dense everywhere. This unique limit is defined to be *the length of the arc of the circle* with which we started. This definition is allowable, since the length, so defined, clearly has the property that *the length of the sum of two adjacent arcs is the sum of the lengths of those two arcs*, and also the property that *if an arc is the unique limit of a sequence of arcs, its length is the limit of their lengths.*

It then follows, by the same argument as before, that, *when the angle subtended at the center by the arc under consideration is kept unaltered, and the radius of the circle is diminished indefinitely, the ratio of the length of the arc to the radius diminishes, and has therefore a unique limit.*

Calling this limit θ, it follows by the same argument as before, that, if the angle subtended at the center by the arc of the circle considered be m/p of two right angles, θ is the unique limit of $mn \tan a_{np}$. Hence,

$$\frac{\theta}{\pi} = \frac{\underset{n \,=\, \infty}{\text{limit}}\ mn \tan a_{np}}{\underset{n \,=\, \infty}{\text{limit}}\ np \tan a_{np}} = \frac{m}{n} = \textit{ratio of the angle to two right angles.}$$

Again, if the ratio of the angle to two right angles is irrational, we can, corresponding to each value of the integer p, assign an integer m, such that the angle lies between $m\, a_p$ and $(m+1)\, a_p$. Hence, by the same argument,

$$\frac{m}{p} < \frac{\theta}{\pi} < \frac{m+1}{p}$$

for each value of p, so that,

$$\frac{\theta}{\pi} = \underset{p \,=\, \infty}{\text{limit}}\ \frac{m}{p} = \textit{ratio of the angle to two right angles.}$$

Hence, we may properly take θ as the measure of the angle, π being taken to be that of two right angles. The unit of angle will then be the angle such that the arc of a circle subtending that angle at the center bears to the radius a ratio which, as the radius diminishes indefinitely, itself diminishes, and has unity as limit. This angle is called *a radian*, and the system of measurement of angles here adopted is called *circular measure*.

§ 38. *Unique Limits of* $\dfrac{\sin \theta}{\theta}$ *and* $\dfrac{1-\cos \theta}{\theta}$.
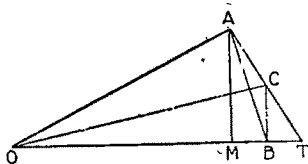
Fig. 11.

Let AB be a chord of a circle subtending a small acute angle at the center O. Let AM be the perpendicular from A on OB, AT be the tangent at A, meeting OB in T, and OC the line through O bisecting the chord AB, and meeting AT in C. Then, as we saw, the arc AB lies between the lengths of AB and $AC + CB$. But since the angles AMB and CBT are right angles,

$$AM < AB \text{ and } CB < CT;$$

therefore,

$$AM < \text{arc } AB < AT.$$

Dividing throughout by the radius, and then letting the radius diminish in definitely,

$$\sin \theta < \theta < \tan \theta,$$

θ being the circular measure of the angle AOB. Dividing through by $\sin \theta$,

$$1 < \theta/\sin \theta < \sec \theta.$$

Letting the angle AOB decrease without limit, $\sec \theta$ has the limit unity; therefore, the same is the unique limit of $\theta/\sin \theta$, or of $\sin \theta/\theta$.

Again, using the fundamental relation, to be proved as (3) in § 40,

$$\frac{\theta^2}{\sin^2 \theta} = \frac{\theta^2}{(1-\cos \theta)(1+\cos \theta)}$$

has the limit unity, and the second factor in the denominator has the limit 2. Hence,

$$\frac{1-\cos \theta}{\theta}$$

has the limit zero.

§§ 39–46.   *Formulae for the Sine and Cosine of the Sum and Difference*
of Two Angles.
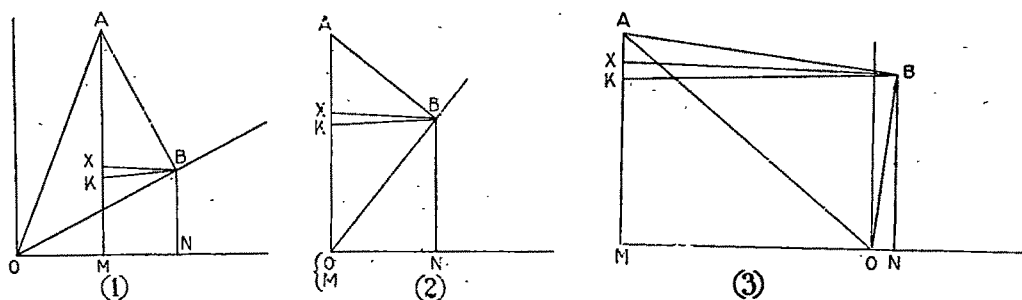
FIGS. 12.

§ 39.   Let *BON* and *BOA* be acute angles (Fig. 12 (1), (2), (3)) on opposite
sides of *OB*, and let the angles *ABO*, *BNO* be made right angles.   Also draw
AM perpendicular to *ON* and *BK* to *AM*, and make *MX* = *BN*.   Then we know
(§ 2) that $KM < XM$, and the angle *BXM* is equal to the angle *XBN*, and is
acute.

When the point *A* moves up to *O* as limiting point, so do *B*, *K*, *M* and *N*.
Consequently (§ 1) the angles of the quadrilateral *BNMK* together have four
right angles for their limit, and therefore the angle *KBN* has a right angle for
limit.   The same is therefore true of the acute angle *XBN*.   Consequently, the
angle *XBK* has zero as limit.

For the same reason, the sum of the angles of the triangle *BON* has two
right angles as limit, so that the sum of the angles *OBN* and *BON* has a right
angle for limit.   Since *KBO* and *OBN* make up *KBN*, which, as we saw, has a
right angle for limit, it follows that *KBO* has *BON* as limit.   Similarly,
KAB + *ABK* has a right angle for limit; whence *KAB* and *KBO* have the
same limit, viz., as we saw, the angle *BON*.   Now

$$AM = AK + KM = AK + XM - KX = AK + BN - KX;$$

therefore,

$$\frac{AM}{OA} = \frac{AK}{AB}\frac{AB}{OA} + \frac{BN}{OB}\frac{OB}{OA} - \frac{KX}{KB}\frac{KB}{AB}\frac{AB}{OA}.$$

Since the ratios converge uniformly to their limits, we get, letting *A* move up
to *O* as limit,

$$\sin AOM = \cos BON \sin AOB + \sin BON \cos AOB. \qquad (a)$$

Writing a for the angle AOB, and b for the complement of the angle BON, we get, by §§ 22 and 25,

$$\cos(a - b) = \cos a \cos b + \sin a \sin b. \tag{1}$$

Again,

$$\frac{OM}{OA} = \frac{ON - MN}{OA} = \frac{ON}{OB} \cdot \frac{OB}{OA} - \frac{MN}{BK} \cdot \frac{BK}{AB} \cdot \frac{AB}{OA}.$$

But, as before KX/KB converged uniformly to zero, and therefore KM/BN to unity, so now MN/BK has the unique limit unity. Hence, proceeding to the limit in the last equation,

$$\cos AOM = \cos BON \cos AOB - \sin BON \sin AOB, \tag{\beta}$$

that is, with the same notation as before,

$$\sin(a - b) = \sin a \cos b - \cos a \sin b. \tag{2}$$

§ 40. The above formulæ for $\cos(a - b)$ and $\sin(a - b)$ have been proved on the assumption that a and b are both acute. The case when $b = a$ can be considered as the limiting case of a sequence of cases in which b approaches a as limit, since the sine and cosine have been shown to be continuous. In this case also the equations must therefore hold. The sine equation then reduces to an identity, but the cosine equation gives us the fundamental relation

$$1 = \cos^2 a + \sin^2 a, \tag{3}$$

whence also, dividing by $\tan^2 a$,

$$\sec^2 a = 1 + \tan^2 a. \tag{4}$$

Thus the equations giving the sine and cosine of $(a - b)$ hold in all cases when a and b are acute angles.

§ 41. Now if in (α) and (β) we put b for the angle BON itself, instead of for its complement, we have the formulæ

$$\sin(a + b) = \sin a \cos b + \cos a \sin b, \tag{5}$$
$$\cos(a + b) = \cos a \cos b - \sin a \sin b, \tag{6}$$

provided a and b are acute.

§ 42. If we change b into $-b$ the formulæ (1) and (6) are interchanged, and so are (2) and (5); therefore we may consider (5) and (6) only, as including (1) and (2), a being a positive acute angle and b a positive or negative acute angle. If we change a into $-a$, however, the formula (5) becomes (2), and (6) becomes (1); hence, (5) and (6) hold when either a or b, or both, are positive or negative acute angles.

Again, if we add $\frac{\pi}{2}$ to a, or to b, (5) and (6) are interchanged, and are therefore still true. Since these operations together with the preceding ones may be carried out any number of times, this shows that the formulæ (5) and (6) hold for any positive or negative values of a and b, including those which are multiples of a right angle, since the formulæ clearly hold for zero values of a or b, or both.

§ 43.   Putting $b = a$, we get

$$\sin 2\,a = 2 \sin a \cos a,$$
$$\cos 2\,a = \cos^2 a - \sin^2 a = 2 \cos^2 a - 1 = 1 - 2 \sin^2 a.$$

§ 44.   Hence,

$$2 \sin^2 \tfrac{1}{2}\, a = 1 - \cos a,$$
$$2 \sin \tfrac{1}{2}\, a \cos \tfrac{1}{2}\, a = \sin a,$$

whence

$$\tan \tfrac{1}{2}\, a = (1 - \cos a)/\sin a.$$

Applying this to the case when a is a right angle,

$$\tan \frac{\pi}{4} = 1, \quad \sin \frac{\pi}{4} = \cos \frac{\pi}{4} = 1/\sqrt{2}.$$

§ 45.   Putting $b = 2\,a$, we get

$$\sin 3\,a = 3 \sin a - 4 \sin^3 a,$$
$$\cos 3\,a = 4 \cos^3 a - 3 \cos a.$$

§ 46.   These equations may be used to give equations for the sine and cosine of $\frac{a}{3}$. In particular, putting $3a = \pi/2$, the second equation gives, after dividing by $\cos^3 a$,

$$0 = 4 - 3 \sec^2 a = 1 - 3 \tan^2 a,$$

so that

$$\tan \frac{\pi}{6} = 1/\sqrt{3}, \quad \sin \frac{\pi}{6} = \frac{1}{2}, \quad \cos \frac{\pi}{6} = \frac{\sqrt{3}}{2}.$$

Hence, by the formulæ for the complementary angle,

$$\tan \frac{\pi}{3} = \sqrt{3}, \quad \sin \frac{\pi}{3} = \frac{\sqrt{3}}{2}, \quad \cos \frac{\pi}{3} = \tfrac{1}{2}.$$

§ 47.   *Differential Coefficients of the Sine, Cosine and Tangent.*

From (5) we have

$$\frac{\sin (a + h) - \sin a}{h} = \sin a \, \frac{\cos h - 1}{h} + \cos a \, \frac{\sin h}{h},$$

whence, if we are using circular measure, by § 38, the left-hand side has, as h approaches zero in any manner, a unique limit, viz., $\cos a$. Thus $\sin a$ is a differentiable function and

$$\frac{d}{da} \sin a = \cos a;$$

similarly from (6),

$$\frac{d}{da} \cos a = -\sin a;$$

whence also

$$\frac{d}{da} \tan a = \sec^2 a.$$

Hence, it follows that *the sine, cosine and tangent possess all their differential coefficients of every order.*

§§ 48−51.   *Continuity and Differentiability of the Length* $2\pi f(r)$ *of a Circle of Radius* r.

§ 48.   The length of a circle of radius r is evidently a function of r alone; let it be $2\pi f(r)$. Then, by what precedes, $f(r)$ has a unique limit, when r is indefinitely decreased, and this limit is unity. It is almost immediately evident that $f(r)$ is a continuous function of r.   For the length of the circle is the limit of the monotone ascending sequence of inscribed regular polygons, and also of the monotone descending sequence of circumscribed regular polygons. From §§ 7 and 12, where it was shown that the sine-ratio and the cosine-ratio are continuous functions, it follows that the tangent-ratio — viz., that of the perpendicular to the base — is so also, and hence it follows that the perimeters of the circumscribed polygons are continuous functions of the radius. Thus the length of the circle, being the limit both of a monotone ascending and of a monotone descending sequence of continuous functions, is itself a continuous function of the radius.*   Hence, $f(r)$ *is a continuous function of* r.

§ 49.   Using the notation of § 3 and Fig. 2, we had

$$CP - BN < AM - CP,$$

whence, supposing these lengths to be the halves of corresponding sides of regular polygons, with O as common center, whose perimeters are p_B, p_C, p_A,

$$p_C - p_B < p_A - p_C.$$

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* W. H. Young: "On a Test for Continuity," 1908, *Proc. Royal Society of Edinburgh*, Vol. XXVIII, pp. 217–221.

Since this is true for all such polygons, we have, denoting the circumference of the circle through A by c_A, and so for other points,

$$c_C - c_B < c_A - c_C.$$

Now suppose that we have $(n + m)$ equal stretches marked off along the radius, and we have the corresponding inequalities,

$$c_3 - c_2 > c_2 - c_1,$$
$$c_4 - c_3 > c_3 - c_2,$$
$$\cdots \cdots \cdots \cdots \cdots,$$
$$c_{n+m+1} - c_{n+m} > c_{n+m} - c_{n+m-1}.$$

Hence, it follows that

$$c_{n+m+1} - c_{n+1} > c_{n+m} - c_n > c_{n+m-1} - c_{n+m-m} > \cdots > c_{m+1} - c_1.$$
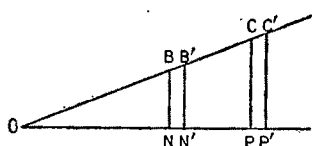
FIG. 13.

Thus if B' lie on the other side of B from O, and $BC = B'C'$, then if BC/CC' is rational, and is in the ratio n/m, we have

$$c_{C'} - c_{B'} > c_C - c_B.$$

Now if BC be fixed and $= h$, the points whose distances from O are rational with respect to h are dense everywhere, and the above shows that the function

$$\frac{f(r + h) - f(r)}{h}$$

is a monotone increasing function of r, with respect to this everywhere-dense set of points. But it is also a continuous function of r (§ 48); therefore it is a monotone increasing continuous function of r.

§ 50. Hence, if, as before, $BC = CA$,

$$\frac{f(OC) - f(OB)}{BC} < \frac{f(OA) - f(OC)}{CA},$$

adding numerators and denominators, we see that the first fraction is

$$< \frac{f(OA) - f(OB)}{OB}.$$

That is,

$$\frac{f(r + h) - f(r)}{h} < \frac{f(r + 2h) - f(r)}{2h}.$$

Hence, by the same reasoning as before, $\dfrac{f(r+h) - f(r)}{h}$ is a monotone increasing continuous function of h, r being constant.

§ 51. From the preceding result it follows that $f(r)$ has a right-hand differential coefficient for every positive value of r. But the incrementary ratio $\dfrac{f(r+h) - f(r)}{h}$ has been shown to be a monotone increasing function of r; therefore, the same is true of its unique limit, viz., the right-hand differential coefficient. Therefore, this right-hand differential coefficient has a unique limit on each side, and that unique limit is therefore its value there, by a known property of derivates, and is the common value of all the derivates there, and the common value of their limits on the two sides. Thus $f(r)$ *possesses a differential coefficient, and it is a monotone increasing continuous function.*

§§ 52–55. *The Defect of a Triangle.*

§ 52. Let ABC be a right-angled triangle, with the angle C a right angle. Divide BC into four equal parts at C_2, C_1 and D_2 (Fig. 14). Let $A_2 C_2$ and $A_1 C_1$ be perpendicular to BC and meet BA in A_2 and A_1 respectively. Also draw the perpendicular $D_2 Q_2$ from D_2 to BC and make it of length equal to $C_2 A_2$. Join BQ_2 and produce it to meet AC in P_2, which it must do, since the point Q_2 obviously lies inside the triangle.
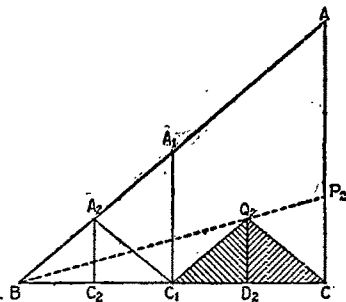
FIG. 14.

Now the triangles $A_2 B C_2$, $A_2 C_2 C_1$, $Q_2 C_1 D_2$, $Q_2 D_2 C$ are all congruent, and therefore have the same *defect*; that is, the difference between two right angles and the sum of the angles of the triangle. Let this defect be d_2, and the defect of the triangle $P_2 BC$ be d'_2. Then, since the two triangles with vertex Q_2 are internal to the triangle $P_2 BC$,

$$2 d_2 < d'_2. \tag{1}$$

Also

$$\tan P_2 BC < Q_2 D_2 / BD_2 < A_2 C_2 / 3 BC_2. \tag{2}$$

Let us next divide into eight, then into sixteen, ... equal parts, and at each stage let us form triangles on the segments between the points of division, congruent to the first such triangle, viz., $A_n B C_n$, where $A_n C_n$ is perpendicular to BC, and A_n lies on BA. Let Q_n be the vertex of the first of these triangles on the other side of the middle point C_1 from B, so that there are 2^{n-1} of the triangles between C_1 and C. Join BQ_n and produce to meet AC in P_n. Then, denoting the defects of the triangles $A_n B C_n$ and $P_n B C$ by d_n and d'_n, we have two relations at each stage corresponding to (1) and (2); viz.,

$$2^{n-1} d_n < d'_n,$$

$$\tan P_n B C < A_n C_n / (2^{n-1} + 1) B C_n.$$

Proceeding to the limit with the latter relation, we see that $\tan P_n B C$ converges to zero, since $A_n C_n / B C_n$ converges to $\tan ABC$. Hence, the triangle $P_n B C$ folds up into the straight line BC, so that its defect d'_n has the unique limit zero. Hence, by the former of the two relations, $2^{n-1} d_n$ has the unique limit zero, and therefore so has $2^n d_n$. But

$$2^n B C_n = BC;$$

therefore

$$d_n / B C_n \text{ has the unique limit zero.}$$

§ 53. Now let C'' be any point between C_n and C_{n-1}. Draw $C'' A''$ perpendicular to BC, meeting BA in A'', and let d'' be the defect of the triangle $A'' B C''$. Then, since this triangle lies inside the triangle $A_{n-1} B C_{n-1}$,

$$d'' < d_{n-1},$$

and since C'' lies between C_n and C_{n-1},

$$B C'' > \tfrac{1}{2} B C_{n-1}.$$

Hence,

$$d'' / B C'' < \tfrac{1}{2} d_{n-1} / B C_{n-1}.$$

Hence, $d'' / B C''$ has also the unique limit zero as C'' approaches B as limiting point in any manner.

§ 54. In the above discussion the angle ABC was kept fixed. Now let us take any sequence of right-angled triangles, Δ_1, Δ_2, ..., such that one of the sides x_1, x_2, ... containing the right angle decreases with zero as unique limit. Then the acute angle adjacent to x_n has a definite upper bound, less than or equal to a right angle. If this upper bound is less than a right angle, we can draw a fixed angle ABC equal to it, and fit the successive triangles Δ_1, Δ_2, ... into this angle, as for instance $D''BC'''$ in Fig. 15. If $D''C'''$ meet AB in A'', the defect of $D''C''B$ is less than that of $A''C'''B$, and therefore this defect divided by BC''' has zero as unique limit as C''' approaches B as limiting point.
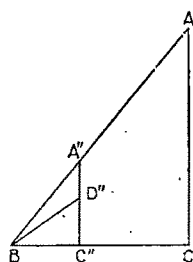
FIG. 15.

If, however, the upper bound is a right angle, it must be the upper limit. Thus if we choose out a sub-sequence of the triangles having other than a right angle as the unique limit of the acute angle in question, the preceding investigation holds. If, however, this acute angle has a right angle as unique limit, the second of the acute angles has the unique limit zero, and has therefore an upper bound which is less than a right angle. Thus, if we know that the hypotenuse, and therefore also the other side containing the right angle, has the limit zero, it follows from the preceding that the limit of the ratio of the defect to that second side, *a fortiori* to the hypotenuse, is zero. Thus in any case, whether or no the angle first in question has the upper limit a right angle, the limit of the defect over the hypotenuse is zero.[*]

Taking the hypotenuse to be of the first order of small quantities, we may therefore say that the defect of a right-angled triangle is of order higher than the first.

§ 55. Since any triangle may be divided into two right-angled triangles, it follows that *if all of the sides of the triangle be considered to be of the first order of small quantities, the defect of the triangle is of order higher than the first.*

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[*] This is clearly true whether or no the hypotenuse has zero as limit. For, if not, the triangle folds up into a straight line and therefore the defect vanishes.

§ 56.   *The Approximate Sine Formulae.*
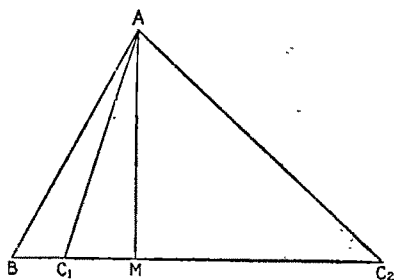
FIG. 16.

Let ABM be a triangle with a right angle at M.   Then, since the defect is of order higher than the first,

$$A\hat{B}M = \frac{\pi}{2} - B\hat{A}M - e_2,$$

or, say,

$$B = \frac{\pi}{2} - A - e_2.$$

Hence,

$$\sin B = \cos (A + e_2) = \cos A \cos e_2 - \sin A \sin e_2 = \cos A - e',$$

where, by § 38, e_2', like e_2, is of order higher than the first.

But, as proved in §§ 9 and 12,

$$\sin B < AM/AB < \cos A.$$

Therefore,

$$0 < \frac{AM}{AB} - \sin B < e_2'.$$

Hence, *AM differs from AB sin B, and similarly from AB cos A, by a quantity which is of an order higher than the second of small quantities.*

Therefore, if ABC is any triangle, whose angles at B and C are neither of them a right angle, and we drop the perpendicular AM on CB, AM differs by a quantity of order higher than the second from each of the expressions

$$c \sin B \text{ and } b \sin C,$$

a, b and c denoting as usual the sides opposite the angles A, B and C respectively. Hence, *neglecting small quantities of order higher than the second,*

$$\frac{\sin B}{b} = \frac{\sin C}{c},$$

and similarly $= \sin A/a$, it being noticed that, by the above, these approximate

equations hold whether or no one of the angles A, B and C is a right angle, in which case the sine is unity.

§ 57. *Approximate Equality of the Arc and Its Chord.*
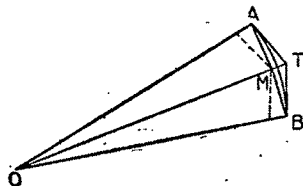
FIG. 17.

Let A and B be neighboring points on a circle of center O, M the middle point of the chord AB, and TA, TB the tangents at A and B, so that OMT is a straight line, and the angles at M are right angles.

Then we know that the arc AB is less than $AT + TB$; therefore, the half arc is less than AT. Hence, also, the half arc of the circle with OM as radius is less than AM. It follows that

arc AB — chord AB < arc AB — arc with AM as radius

$$< (f(OA) - f(OM)) A\hat{O}B < \frac{f(OA) - f(OM)}{OA - OM} TM . A\hat{O}B.$$

Now the incrementary ratio in the last expression has $f'(AO)$ as unique limit, and is therefore finite; $A\hat{O}B$ is of the first order, and TM of order higher than the first, since TM/AM converges uniformly to tan 0, that is, to zero. Thus *the difference between the chord and the arc is of order higher than the second.*

§§ 58, 59. *Killing's Formulae.*
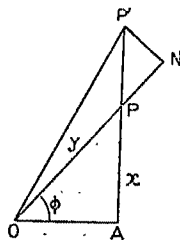
FIG. 18.

§ 58. Let P' be a point near P on a fixed straight line AP (Fig. 18), so that PP' and the angle POP', or $\Delta\phi$, are to be considered as of the first

order of small quantities. Let OA be the initial line from which the angle ϕ, that is AOP, is measured. Draw $P'N$ perpendicular to OP, and let us denote

$$OP \text{ by } y, \quad OP' \text{ by } y + \Delta y,$$
$$AP \text{ by } x, \quad AP' \text{ by } x + \Delta x.$$

Then, by § 54, $P'N$ differs from $f(y + \Delta y) \Delta\phi$ by a small quantity e_{2+} of order higher than the second. Also, as P' moves up towards P, $P'N/PP'$ converges to $\sin \psi$, where ψ is the angle OPA. Hence, $\dfrac{\Delta x}{\Delta\phi}$ converges to $f(y) \operatorname{cosec} \psi$. In other words, x has a differential coefficient with respect to ϕ, and

$$\frac{dx}{d\phi} = f(y) \operatorname{cosec} \psi.$$

Again,

$$\Delta y = OP' - OP = OP' - ON + PN;$$

therefore, since, by § 54, ON differs from $OP' \cos P'ON$ by a small quantity e'_{2+} of order higher than the second,

$$\frac{\Delta y}{\Delta x} = (y + \Delta y) \frac{1 - \cos \Delta\phi}{\Delta\phi} \frac{\Delta\phi}{\Delta x} + \frac{PN}{PP'} + e'_{2+}.$$

The first term here converges to zero, since, by the above, the third as well as the first factor remains bounded, and, by § 38, the second factor converges to zero. Hence, the left-hand side, like the right-hand side, converges to $\cos \psi$. That is, y possesses a differential coefficient with respect to x, and

$$\frac{dy}{dx} = \cos \psi.$$
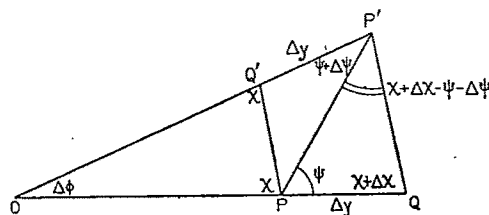
FIG. 19.

§ 59. Now on OP and OP' respectively take Q and Q', so that

$$PQ = Q'P' = \Delta y.$$

Let the equal angles OPQ' and $OQ'P$ be denoted by χ, and OQP' and $OP'Q$

by $\chi + \Delta \chi$. Then, neglecting small quantities of order higher than the first, we get, by considering the triangles $Q'PP'$ and QPP' (Fig. 19),

$$(\pi - \chi) + (\pi - \psi - \chi) + (\psi + \Delta\psi) = \pi,$$
$$\psi + (\chi + \Delta\chi) + (\chi + \Delta\chi - \psi - \Delta\psi) = \pi.$$

Hence,

$$\chi = \frac{\pi}{2} + \tfrac{1}{2}\Delta\psi,$$

$$\chi + \Delta\chi = \frac{\pi}{2} + \tfrac{1}{2}\Delta\psi,$$

and therefore,

$$\pi - \chi - \psi = \chi + \Delta\chi - \psi - \Delta\psi = \frac{\pi}{2} - \psi - \tfrac{1}{2}\Delta\psi;$$

that is,

$$\angle\, Q'PP' = \angle\, PP'Q = \frac{\pi}{2} - \psi - \tfrac{1}{2}\Delta\psi.$$

Therefore, projecting first $Q'P$ and PQ, and then $Q'P'$ and $P'Q$ perpendicular to PP', and equating the results, we get, neglecting small quantities of order higher than the second, by § 56,

$$PQ'\,\sin\,Q'PP' + PQ\,\sin\,\psi = P'Q\,\sin\,PP'Q + Q'P'\,\sin\,(\psi + \Delta\psi),$$

or, remembering that we may replace the chord by the arc, if we are neglecting small quantities of order higher than the second (§ 57),

$$f(y)\,\Delta\phi\,\cos\,(\psi + \tfrac{1}{2}\Delta\psi) + \Delta y\,\sin\,\psi = f(y + \Delta y)\,\Delta\phi\,\cos\,(\psi + \tfrac{1}{2}\Delta\psi)$$
$$+ \Delta y\,\sin\,(\psi + \Delta\psi).$$

Hence, rearranging terms and dividing through by $\Delta y\,\Delta\phi$, so that the neglected terms, which we now re-introduce, contribute a term whose limit is zero,

$$\frac{f(y + \Delta y) - f(y)}{\Delta y} \cdot \cos\,(\psi + \tfrac{1}{2}\Delta\psi)$$
$$= -\frac{\sin\,(\psi + \Delta\psi) - \sin\,\psi}{\Delta\psi} \cdot \frac{\Delta\psi}{\Delta\phi} + e' = \cos\,\psi\,\frac{\Delta\psi}{\Delta\phi} + e,$$

where e' and e have the limit zero.

Since every term in this equation except $\Delta\psi/\Delta\phi$ is known to have a unique limit, when P' moves up to P, it follows that that ratio also has a unique limit, so that ψ has a differential coefficient with respect to ϕ, given by

$$\frac{d\psi}{d\phi} = - f'(y),$$

provided $\cos\,\psi$ is not zero.

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§ 60. *The Sine Formulae.*

We have now fully justified the following equations, obtained first by Killing. In this and the following articles we follow that author, in obtaining, for the convenience of readers, the fundamental formulæ connecting the sides and angles of any triangle.

$$\left. \begin{aligned} \frac{dx}{d\phi} &= f(y) \, \text{cosec} \, \psi, \\ \frac{dy}{dx} &= \cos \psi, \\ \frac{d\psi}{d\phi} &= -f'(y), \end{aligned} \right\} \quad \text{(Killing's equations)}.$$

In other words, assuming that the length OA and the angle OAP are kept constant, we have the ratios of the increments of the remaining four parts of the triangle POA. Thus, in particular,

$$\frac{dy}{d\psi} = -\frac{\cos \psi}{\sin \psi} \cdot \frac{f(y)}{f'(y)},$$

integrating which, we get

$$f(y) \sin \psi = \text{const.}$$

Since as P moves up to A, x approaches zero, y approaches OA, or say c, and ψ approaches $\pi - \beta$, where β is the angle OAP, we have

$$f(y) \sin \psi = f(c) \sin \beta.$$

Hence, in any triangle ABC, with the usual notation for the sides and angles,

$$\frac{f(a)}{\sin A} = \frac{f(b)}{\sin B} = \frac{f(c)}{\sin C}. \qquad \text{(I)}$$

§ 61. *The Cosine Formulae.*

Thus, returning to our previous notation,

$$f(c) \sin \phi = f(x) \sin \psi,$$

whence, differentiating,

$$f(c) \cos \phi \, d\phi = f'(x) \, dx \sin \psi + f(x) \cos \psi \, d\psi,$$

or, replacing the increments by their proportionals, by § 60,

$$f(c) \cos \phi = f'(x) f(y) - f(x) f'(y) \cos \psi.$$

Hence, in any triangle,

$$f(c) \cos A = f'(a) f(b) - f(a) f'(b) \cos C. \qquad \text{(II)}$$

§§ 62, 63. *Conclusion. The Sine and Cosine Formulae in the Three Geometries.*

§ 62. Eliminating cos C between the last equation and the similar one,

$$f(a) \cos C = f'(c) f(b) - f(c) f'(b) \cos A,$$

we get

$$f(c) \cos A = f'(a) f(b) - f'(b) f'(c) f(b) + f(c) (f'(b))^2 \cos A.$$

Therefore,

$$f(c) \cos A \left\{ 1 - (f'(b))^2 \right\} = f(b) \left\{ f'(a) - f'(b) f'(c) \right\}.$$

Hence, interchanging b and c, and eliminating cos A,

$$\frac{1 - [f'(b)]^2}{[f(b)]^2} = \frac{1 - [f'(c)]^2}{[f(c)]^2}.$$

Hence, since b and c are any two stretches which form two of the sides of a triangle, we have

$$\frac{1 - [f'(b)]^2}{[f(b)]^2} = \text{const.} = \frac{1}{k^2}, \text{ say,}$$

for all values of b.

Integrating and remembering that $\underset{b=0}{\text{limit}} \, f(b) = 0$, we have three cases to distinguish:

1) k^2 is positive,

$$\sin^{-1} \frac{f'(b)}{k} = \frac{b}{k},$$

whence

$$f(b) = k \sin \frac{b}{k},$$

$$f'(b) = \cos \frac{b}{k}.$$

2) k^2 is negative,

$$\sinh^{-1} \frac{f'(b)}{k} = \frac{b}{k},$$

$$f(b) = k \sinh \frac{b}{k},$$

$$f'(b) = \cosh \frac{b}{k}.$$

3) k is infinite,

$$f(b) = b.$$

Now in the Lobatchefsky geometry, as we have seen, $f'(b)$ is a monotone increasing function of b; thus case 1) can only be the Riemann geometry, in which $f'(b)$ is a monotone decreasing function of b, for small values of b; case 3), in which $f'(b)$ is constant, is the Euclidian case, and case 2) is the Lobatchefsky geometry.

§ 63. Thus in the Lobatchefsky geometry we have:

$$\frac{\sin A}{\sinh \frac{a}{k}} = \frac{\sin B}{\sinh \frac{b}{k}} = \frac{\sin C}{\sinh \frac{c}{k}}, \qquad \text{(Lobatchefsky I)}$$

$$\sinh \frac{c}{k} \cos A = \cosh \frac{a}{k} \sinh \frac{b}{k} - \sinh \frac{a}{k} \cosh \frac{b}{k} \cos C. \qquad \text{(Lobatchefsky II)}$$

In the Euclidian geometry we have:

$$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}, \qquad \text{(Euclid I)}$$

$$c \cos A = b - a \cos C. \qquad \text{(Euclid II)}$$

Finally, in the Riemann geometry, we have:

$$\frac{\sin A}{\sin \frac{a}{k}} = \frac{\sin B}{\sin \frac{b}{k}} = \frac{\sin C}{\sin \frac{c}{k}}, \qquad \text{(Riemann I)}$$

$$\sin \frac{c}{k} \cos A = \cos \frac{a}{k} \sin \frac{b}{k} - \sin \frac{a}{k} \cos \frac{b}{k} \cos C. \qquad \text{(Riemann II)}$$

From these formulæ all the others can be obtained.

Curves in Non-Metrical Analysis Situs with an Application in the Calculus of Variations.*

By N. J. Lennes.

§ 1. *Introduction.*

This paper contains a body of definitions and theorems relating to simple curves, limit-curves, etc., which, it is hoped, will be of general usefulness in a considerable range of non-metrical investigations in analysis situs and related subjects. It had its origin in an attempt to prove certain theorems concerning the existence of solutions in the calculus of variations. Indeed, the definition of an arc given in § 4 is merely an enumeration of those properties of a certain set of limit-points of a sequence of arcs which appear when one attempts to prove directly that they constitute a continuous arc.

It is apparent that in a geometry possessing linear order and continuity curves and limit curves exist entirely independently of metric properties. Hence the discussion so far as it relates to these is carried out on non-metric hypothesis. Schoenflies testifies to the desirability of this procedure in the following words (after quoting Cantor's definition of "Zusammenhang," which is stated in terms of equality of segments) : "Wenn nun auch der Abstand zweier Punkte für die hier vorliegenden Untersuchungen einen axiomatischen geometrischen Grundbegriff bildet, so scheint es mir doch zweckmässig, rein mengentheoretische Definitionen überall da zu bevorzugen, wo es möglich ist, ..." In spite of this explicit expression of preference for non-metric treatment "wo es möglich ist," Schoenflies uses metric hypothesis in the proof of practically every important theorem dealing with curves and the regions defined by them.

The argumentation in various parts of the paper requires a considerable body of theorems on simple finite and infinite polygons. Consequently § 2 is

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devoted to polygons. A number of theorems on the finite polygon are proved by the writer in a paper in this Journal.[*] Schoenflies proves the main theorem of § 3; viz., that an infinite continuous simple polygon separates the plane into two connected sets.[†] His treatment, however, makes use of full metric properties as well as the axiom of parallels, and also makes use, without proof, of the theorem for the finite case as stated by Hilbert and Veblen.[‡] The latter theorem, however, had been proved earlier by Schoenflies from metric hypothesis.[§] While the axiomatic basis for this treatment of the infinite polygon is thus considerably weaker than that used by Schoenflies, it is believed that the treatment is shorter than his, while at the same time less is left to be supplied by the reader.

Section 3 deals with approach to limit-points. To prove that there exists a sequence of points on a line approaching a given point as a limit-point, a mild form of continuity is used (see p. 305). This axiom may be regarded as the projective geometry analogue of the Archimedean axiom of metric geometry. So far as known to the writer the axiom in this form is new. The existence of a sequence of sets of points closing down *uniformly* upon a given closed set of points follows immediately without further axioms. The existence of such sequences is fundamental in the discussions that follow, and it is believed they will be generally useful in work on non-metric analysis situs. Indeed, it seems that metric properties have been brought in precisely at that point in the argument where such sequences of sets are here used, and that even by those who have avoided the use of metric properties most consciously and most successfully. Compare for instance Veblen's "Curves in Non-Metrical Analysis Situs"[||] with p. 312 of this paper. The given closed set upon which these sequences of sets of points close down is identical with the *generalized inner limiting set* of Young.[¶] The *uniformity* of approach, however, is peculiar to this paper and it is this *uniformity* which is of importance in the argument.

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[*] Lennes: "Theorems on the Finite Polygon and Polyhedron," Vol. XXXIII (1911), pp. 37–62. For other non-metrical proofs of some of these theorems, see O. Veblen, *Transactions of the American Mathematical Society*, Vol. V (1904), p. 343; and Hans Hahn, *Monatshefte für Mathematik und Physik*, Vol. XIX, pp. 289–303.

[†] Schoenflies: "Beiträge zur Mengenlehre," I, *Mathematische Annalen*, Vol. LVIII, pp. 195–234.

[‡] Hilbert: "Gründlagen der Geometrie" (2d edition, p. 6); and Veblen, *loco citato*, p. 365.

[§] *Gött. Nachr.*, 1902, pp. 185–192.

[||] *Transactions of the American Mathematical Society*, Vol. VI, p. 83.

[¶] W. H. Young and Grace Chisholm Young: "The Theory of Sets of Points," *Cambridge University Press*, p. 69 *et seq.*

The construction used in § 3 to obtain a sequence approaching a given point as a limit is that given by Von Staudt * in his proof of the fundamental theorem of projective geometry. The axiom used in this paper is of course weaker than the full continuity used by Klein † to validate the argument of Von Staudt.

Section 4 contains the definition of arc (or curve) and a proof that it is an arc of a Jordan curve when the definition of the latter is couched in non-metric terms. Various point-set definitions of arcs (or curves) have been given. Veblen ‡ defines "curve" in terms of "point" and "order" and proves that the result is a Jordan curve. However, metric properties are used at one step in showing that the curve is actually a Jordan curve, — a result obtained in this paper by means of uniformity of approach. Schoenfies § defines "curve" as a frontier or outer rim of a connected region having the property of being accessible (p. 312) at every point both from exterior and interior points. Thus the Schoenflies definition of closed continuous curve is analogous to the Dedekind "Schnitt" on the line.

Young ‖ defines a curved arc as "a plane set of points, dense nowhere in the plane, such that, given any norm e, and describing around each point of the set a region of span less than e, these regions generate a single region Re, whose span does not decrease indefinitely with e." In Young's treatment free use is made of metric properties.

In this paper an arc is defined (p. 308) as follows: "*A closed, bounded, connected set of points containing A and B, $A \neq B$, which contains no proper connected subset containing A and B, is a continuous arc whose end-points are A and B.*" This definition seems to be very near the obvious intuitional meaning of the term "arc" or "curve." It has the two properties of "connectedness" and "thinness"; viz., an arc consists of "one piece" and is so "thin," everywhere, that removing any one point, other than an end-point, separates it into two parts.

In section 5 the frontier of a region is considered as a Jordan curve. A proof is given of the theorem of Schoenflies that any outer rim of a connected

* Von Staudt: "Geometrie der Lage," p. 50.

† *Mathematische Annalen,* Vol. VI, p. 139.

‡ Veblen: "Curves in Non-metrical Analysis Situs," *Transactions of the American Mathematical Society,* Vol. VI, pp. 83–98.

§ Schoenflies, *loco citato,* p. 195.

‖ W. H. Young and Grace Chisholm Young, *loco citato,* p. 206. Also W. H. Young: *Quarterly Journal of Pure and Applied Mathematics,* Vol. XXVII, pp. 1–35.

region is a Jordan curve in case it is accessible at every point both from exterior and interior points. It is also shown that an outer rim every point of which is accessible from an exterior point separates the remaining points of the plane into two connected sets, and also that a rim may. be accessible at every point from exterior points·and fail to be accessible from interior points, and hence need not be·a Jordan curve.

Section 6 is devoted to a simple non-metrical proof of the classical theorem that a Jordan curve separates a plane into two connected sets. Section 7 is concerned with limit-arcs of a set of arcs. It is shown that under a certain uniformity condition on the continuity of the set of arcs at least one limit-curve exists.

In section 8 the general theory of the paper is applied to the problem of proving the existence of minimizing curves in an important class of problems in the calculus of variation.

The argumentation is based specifically on the axioms of Professor Veblen.[*]

The undefined symbols of Veblen's axioms are *point* and *order*. He defines a line containing the points A and B as consisting of the points A and B together with all points X which have one of the orders XAB, AXB and ABX. The points X such that the order AXB exists constitute the segment AB. If the points A, B, C are not collinear, the segments AB, BC, CA, together with the points A, B, C, form a triangle, and all points collinear with two points. of a fixed triangle form a plane.

The following axioms are used:

AXIOM I.[†]   *There exist at least three points.*

AXIOM II.   *If the points A, B, C are in the order ABC, they are in the order CBA.*

AXIOM III.   *If the points A, B, C are in the order ABC, they are not in the order BCA.*

AXIOM IV.   *If the points A, B, C are in the order ABC, then A is distinct from C.*

AXIOM V.   *If A and B are two distinct points, there exists a point C such that A, B, C are in the order ABC.*

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* Oswald Veblen: "A System of Axioms for Geometry," *Transactions of the American Mathematical Society*, Vol. V (1904), pp. 345–384.

† The Roman numeral indicates the number of the axiom in Veblen's set.

AXIOM VI. *If the points C and D (C $\neq$ D) lie on the line AB, then A lies on the line CD.*

AXIOM VII. *If there exist three distinct points, there exist three points A, B, C not in any of the orders ABC, BCA, or CAB.*

AXIOM VIII (the Triangle Transversal Axiom). *If three distinct points A, B, C do not lie in the same line, and D and E are points in the orders BCD and CEA, then a point F exists in the order AFB such that D, E, F lie on the same line.*

AXIOM C (Axiom of Continuity).[*] *If all points of a line are divided into two sets such that no point of either set lies between points of the other, then there is one point on the line which does not lie between points of either set.*

The theorems of section 2 are proved by means of Axioms I–VIII. The other theorems are proved by means of Axioms I–VIII and C. We adopt Veblen's definition of segment, line, triangle and plane. The discussion throughout the paper is confined to the plane, and the axioms selected from Veblen's set are plane axioms.

§ 2. *The Simple Polygon.*

The main topic of section 2 is the infinite continuous polygon. Theorems on the finite polygon that are used in the argumentation are inserted for convenience of reference. These theorems were proved explicitly in a paper in the present volume of this Journal.[†] The references to that paper are by page numbers and the number of the proposition; as (28), p. 45. References to propositions in the present paper are by the numbers of the proposition and section only; as (1), § 2. In some cases where the proofs are entirely obvious no reference is made.

DEFINITIONS. *A set of points [X][‡] such that one of the orders AXB and ABX exists, together with the points A and B, forms a "half-line" AB (not a half-line BA). The half-line is said to proceed from A.*

The points lying on two half-lines proceeding from the same point but not lying in the same line form an "angle."

The symbols $\angle$ and $\triangle$ are used for angle and triangle in the usual manner.

A point P is an "*interior point*" of a set if there is a triangle t of which P is an interior point such that every interior point of t (*possibly except* P) is a point of the set.

A set of points is "*entirely open*" if every one of its points is an interior point of the set.

1. THEOREM. *Any line of a plane separates the remaining points of the plane into two entirely open sets such that a segment connecting two points of the same set contains no point of the line, while a segment connecting points of different sets contains a point of the line.*

(For proof see E. H. Moore: "On the Projective Axioms of Geometry," *Transactions of the American Mathematical Society*, Vol. III (1902), pp. 142–158, or Veblen, *loco citato*.)

2. THEOREM. *An angle (triangle) separates the remaining points of the plane in which it lies into two entirely open sets, an interior and an exterior, such that a segment connecting an interior and an exterior point contains one point of the angle (triangle), a segment connecting two interior points contains no point of the angle (triangle) and a segment connecting two exterior points and not containing a vertex contains two or no points of the angle (triangle).*

(For proof see same as preceding.)

DEFINITIONS. *The points lying on a set of segments A_1A_2, A_2A_3, ..., $A_{n-1}A_n$, together with the points A_1, A_2, ..., A_n (called vertices), constitute a finite broken line.*

The point L is said to be an end-point or limit-point of the infinite set of segments A_1A_2, A_2A_3, ..., A_nA_{n+1}, ..., if for every triangle t of which L is an interior point there is a number N (depending on the triangle t) such that for every $n > N$ the segment A_nA_{n+1} lies entirely within the triangle. The set of segments form an "infinite broken line" connecting its end-points A_1 and L. If a point C is connected with L or A_1 by means of a finite or infinite broken line, then the two broken lines together form a broken line connecting A_1 and C or L and C. Such points as A_1, L, C are vertices of the broken line. A segment including its end-points is a special case of a broken line.

Hereafter the expression "broken line" will be used for both finite and infinite broken lines. The word "finite" or "infinite" will be used when we wish to specify particularly the one or the other.

If no point of a broken line is common to two of its segments, a segment and a

vertex, or two vertices (*except possibly the end-points*), the broken line is a "*simple*" broken line.

If a simple broken line connects two points *A* and *B*, and if these points are the same point, the broken line forms a "*simple polygon*." If the broken line is finite, the polygon is "*finite*"; and if the broken line is infinite, the polygon is "*infinite*." The segments of the broken line are the "*sides*" of the polygon, and the vertices of the broken line are the "*vertices of the polygon*."

If a vertex is a limit-point of an infinite sequence of segments, the polygon is said to be "*infinite*" at this vertex or to have a "*limit-point*" at this vertex.

The word "polygon" will be used for both finite and infinite simple polygons.

·An entirely open set of points is said to be *connected* (see note, p. 303) *if for any two points of the set there is a broken line connecting them which lies entirely in the set.*

3. THEOREM. *If A and B are points of an entirely open connected set, then there is a finite broken line connecting them which lies entirely in the set.*

·PROOF. By hypothesis there is a broken line (finite or infinite) in the set connecting the points *A* and *B*. Suppose the broken line is infinite and has just one limit-point *L*. Since *L* lies within the set, there is a triangle *t* containing *L* as an interior point all of whose interior points are points of the set. If *A* is exterior to *t*, trace the given broken line from *A* to a point on *t* and likewise from *B* to a point on *t*. These two finite broken lines, together with a segment connecting end-points of them within *t*, form the required broken line connecting *A* and *B*. Since the broken line has only a finite number of vertices which are limit-points, it follows that a repetition of this construction gives the required broken line for the general case.

DEFINITIONS. *A set of points* [*P*] *is said to* "*separate*" *the remaining points of the plane into two sets if every broken line connecting a point in one set with a point in the other contains at least one point of* [*P*].*

If a triangle t_i is constructed about each vertex L_i of a set of broken lines [*b*], then the segments of [t_i], together with those segments of [*b*] which are partly or entirely exterior to every triangle of [t_i], are called the "*exposed*" set of [*b*] with respect to [t_i].

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* The following is a more general definition of *separation*: "·A set of points [*P*] is said to separate a connected set [*R*] if the points of [*R*] not in [*P*] do not form a connected set," the term *connected set* being used in the sense of § 3. See page 303.

4. Theorem. *If* [b] *is a finite set of broken lines, the remaining points of the plane form an entirely open set.*

Proof. Let L_1, L_2, ..., L_n, or $[L_i]$, be the limit-vertices of [b] and P any point not of [b]. By (17), p. 41, there is for each point L_i a triangle t_i of which P is an exterior point. By the definition of "continuous broken line" there is only a finite number of exposed segments of [b] with respect to $[t_i]$. Hence by (17), p. 41, there is a triangle t of which P is interior, and every exposed segment, together with its limit-points, exterior. Then, by (16), p. 41, there is no point of [b] within t.

5. Theorem. *If* [b] *is a finite set of broken lines and ABC any angle, B not a limit-vertex of one of the broken lines, then there is a ray BK within the angle ABC which contains no vertex of* [b].

Proof. If there are n limit-vertices of [b] on or within $\angle ABC$, construct rays from B within the angle forming $2n + 1$ angles of which no two have an interior point in common (8), p. 40. Then there is at least one angle of this set such that there is no limit-vertex of [b] on or within it. Hence, by (2), § 2, and the definition of broken line, there are only a finite number of vertices of [b] within this angle; and hence, by (8), p. 40, the required ray BK may be constructed within it.

6. Theorem. *If* [b] *is any finite set of broken lines and ABC an angle such that there is no point of* [b] *on the segments AB and BC or their end-points, then there is a point C' on BC such that there is no point of* [b] *on or within the triangle ABC'.*

Proof. About each limit-vertex L_i of [b] on or within $\angle ABC$ construct a triangle t_i such that no point of the segments AB and BC or their end-points lies on or within a triangle t_i. Then there is only a finite number of exposed segments within $\angle ABC$, and hence, by (15), p. 41, there is a point C' on BC such that there is no point of an exposed segment on or within the triangle ABC', and hence no point of [b] on or within this triangle.

7. Theorem. *If* t_1 *is a triangle enclosing a limit-vertex L of a set of broken lines* [b], *then there is a triangle* t_2 *also enclosing L, which lies entirely within* t_1 *and on which lies no vertex of* [b]. *An infinite broken line connecting a point* A_1 *exterior to* t_2 *with its only limit-vertex L within* t_2 *meets* t_2 *in an odd number of points.*

Proof. Let ABC be the triangle t_1 enclosing L. Using (5), construct CD and CE so that no vertex of [b] lies on these segments while L is within the

angle DCE. Similarly construct DF and DG and then FH, thus obtaining the triangle FGH which has the required properties. That an infinite broken line connecting an exterior point A_1 with its only limit-point L within t_2 meets t_2 in an odd number of points is then an obvious corollary of (2) and the definition of continuous broken line.
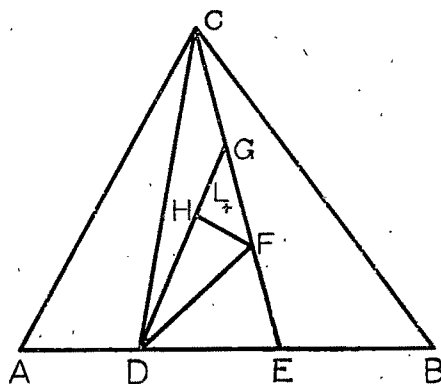

Fig. 1.

8. THEOREM. *If a line contains no vertex of a polygon, or if of an angle and a polygon neither contains a vertex of the other, then such line or angle contains an even number of points of the polygon, zero being an even number.*

PROOF. For the case when the polygon is finite, see (18), p. 42. In case it is infinite, proceed as follows: Let l denote the given line and L_1, L_2, ..., L_n, or $[L_i]$, be the limit-vertices of the polygon, the notation being so arranged that the points are in that order on the polygon which is indicated by the subscripts. Let $[t_i]$ be a set of triangles such that L_i lies within t_i while every point of l is exterior to every triangle of $[t_i]$ (see (1)). Consider the broken line $L_1 L_2$ which consists of two broken lines $A_1 A_2$, $A_2 A_3$, ..., $A_n A_{n+1}$, ..., L_1 and $A_1 A_2'$, $A_2' A_3'$, ..., $A_m' A_{m+1}'$, ..., L_2. By definition (p. 292) there is an N such that, for $n > N$, $A_n A_{n+1}$ lies within t_1, and an M such that, for $m > M$, $A_m' A_{m+1}'$ lies within t_2. Then every point of the broken line $L_1 L_2$ which lies on l is on the finite broken line $A_m' A_{m-1}'$, ..., $A_{n-1} A_n$. As an immediate consequence of (1), this broken line contains an even or odd number of points on l according as A_n and A_m' lie on the same or opposite sides of the line Since L_1 and A_n, and L_2 and A_m' are on the same side respectively of l, it follows that the broken line $L_1 L_2$ contains an even or odd number of points of l according as L_1 and L_2 are on the same or opposite sides of l. The theorem now follows

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exactly as in the case of the finite polygon. The argument for the angle is identical with that given for the line, except that (2) is used instead of (1).

We now define as in the case of the finite polygon.

DEFINITION. *A point not on a polygon is an interior point of the polygon if a half-line proceeding from it and containing no vertex of polygon contains an odd number of points of the polygon. The point is exterior if such half-line contains an even number of points of the polygon.*

9. THEOREM. *If a broken line contains no point of a polygon, it is either entirely exterior or entirely interior.*

PROOF. For the case when the polygon is finite, see (19), p. 43. It remains to make the proof in case the polygon has one or more limit-vertices. We show first that a segment which does not meet the polygon is entirely interior or entirely exterior.

Let A be any point of such segment $A_1 A_2$ or its end-point A_1. Denote by $[L_i]$ the set of limit-vertices of the polygon. About each point L_i construct a triangle t_i of which the segment $A_1 A_2$ is entirely exterior. Construct a half-line AK not meeting a vertex of the polygon (5). Then by (6) there is a point B on AK such that there is no point of the polygon on or within the triangle ABA_2. Again, by (5) there is a ray $A_2 H$ within $\angle A A_2 B$ which contains no vertex of the polygon. Let the ray $A_2 H$ meet the segment AB in R ((6), p. 39). Since the rays AR and $A_2 R$ contain no vertices of the polygon, and the segments AR and $A_2 R$ or their end-points contain no points of the polygon, it follows from the definition of exterior and interior points that the points on these segments, including their end-points, are all exterior or all interior; that is, A and A_2 are both exterior or both interior. But A is any point of the segment $A_1 A_2$, or possibly A_1, and hence the points of this segment, including its end-points, are all interior or all exterior. It now follows immediately that any finite broken line which fails to meet the polygon is all interior or all exterior.

Consider now an infinite broken line $A_1 A_2$, $A_2 A_3$, $\ldots$, $A_n A_{n+1}$, $\ldots$ with a limit-vertex L which does not meet the polygon. Then, by the preceding, the points of this broken line, except L, are all interior or all exterior. Since L does not lie on the polygon, there is by (4) a triangle t containing L as an interior point within which there is no point of the polygon. Connect L with some point K of the broken line $A_1 L$ within t. Then we have a finite broken line connecting A_1 and L, and hence L is interior or exterior according as the remainder of the broken line is interior or exterior.

10. THEOREM. *If P is a point of a side $A_1 A_2$ of a polygon, and if segments PB and PC lie on opposite sides of the line $A_1 A_2$ and contain no point of the polygon, then one segment is entirely exterior and the other entirely interior.*

PROOF. Through P construct a line l such that one ray PK on it does not contain a vertex, (5). Let B' and C' be points on l in the order $B'PC'$ such that B and B' lie on the same side of the line $A_1 A_2$, and such that there is no point of the polygon on $B'C'$ except the point P. Then, by definition, one of the points B' and C' is interior and the other is exterior. Since B and B' are on
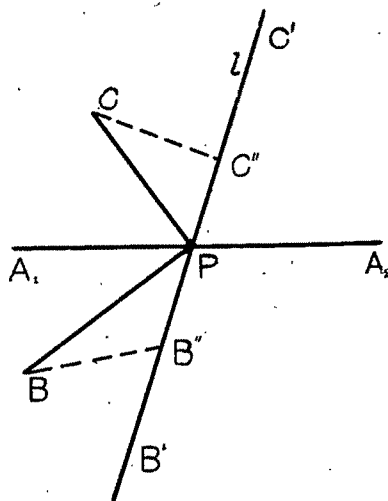
FIG. 2.

the same side of the line $A_1 A_2$, there is no point of the segment $A_1 A_2$ within the angle BPB'. Hence, by (6) there is a point B'' on $B'P$ such that there is no point of the polygon (except P) on or within the triangle BPB''. Hence, by (9) all points of the segment BP are exterior or interior according as B' is exterior or interior. In the same manner we show that the segment CP is exterior or interior according as C' is exterior or interior. Hence one of the segments BP and CP is entirely interior and the other entirely exterior.

It follows also from this argument that:

11. THEOREM. *If two segments AB and AC are both interior or both exterior and have the common end-point A on a side of the polygon, then there is a broken line connecting B and C which does not meet the polygon.*

12. THEOREM. *If two points P and Q lie on the same or different sides of a polygon p, then there is a finite broken line connecting them which is entirely interior, and another which is entirely exterior.*

PROOF. Suppose P and Q lie on two consecutive sides $A_1 A_2$ and $A_2 A_3$ respectively. Then there is a point H on the line $A_3 A_2$ in the order $A_3 A_2 H$ such that there is no point of p on $A_2 H$. Then by (6) there is a point M on $A_2 H$ such that there is no point of p on PM. In the same manner we obtain a point N on the line PM in the order PMN such that there is no point of p on QN. Hence there is no point of p on the broken line PN, NQ.
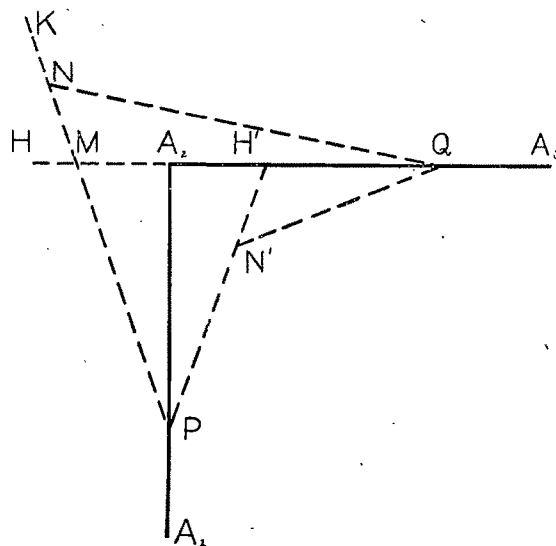
FIG. 3.

In the same manner we find H' on the segment QA_2 and a point N' on the segment PH' such that there is no point of p on the broken line PN', $N'Q$. By (10) and (9) one of these broken lines is interior and the other exterior. By repeating this process and using (11) we now prove the theorem for the case where P and Q are connected by a finite broken line of the polygon.

We next consider the case where the points P and Q are connected by a broken line of p which contains only one limit-vertex L. According to (7) enclose L by a triangle on which lies no vertex of p and no point of p except of the broken line connecting P and Q. Then by (7) each of the broken lines PL and QL meets t in an odd number of points. Let $R_1 R_2, \ldots, R_n$ be the points in which it meets PL as they appear in order on the triangle t. Then

there are two consecutive points, as R_1, R_2, on t between which QL meets t in an odd number of points. Let these points in their order be Q_1, Q_2, ..., Q_m. Then by (10) one of the segments $R_1 Q_1$ and $Q_m R_2$ is interior and the other is exterior. Suppose $R_1 Q_1$ exterior. Then, by the finite case of the theorem and (11), there are finite broken lines connecting both P and Q with points on $R_1 Q_1$ which are entirely exterior. These two broken lines, together with a segment of $R_1 Q_1$, form a finite broken line connecting P and Q which is entirely exterior.

In the same manner, using the interior segment $Q_m R_2$, we obtain an interior broken line connecting P and Q.

13. THEOREM. *A polygon separates the remaining points of the plane into two entirely open connected sets, one consisting of the interior points and the other of the exterior points of the polygon.*

PROOF. (a) By (9) a broken line connecting an interior and an exterior point meets the polygon.

(b) Any two interior points are connected by a broken line which does not meet the polygon. Let M and N be any two interior points. Connect these with points P and Q on the polygon by means of segments MP and NQ which contain no point of the polygon. Then, by (12), M and N may be connected. If M and N are both exterior points we proceed in the same manner.

DEFINITIONS. *The sets $[O']$ and $[O'']$ are complementary subsets of the set $[O]$ if* (a) *the sets $[O']$ and $[O'']$ have no element in common,* (b) *every element of either $[O']$ or $[O'']$ is an element of the set $[O]$,* (c) *every element of $[O]$ is an element of either $[O']$ or $[O'']$.*

A set of points is bounded if there exists a polygon of which every point of the set is an interior point.

A point is an interior point of a set of polygons if it is an interior point of one polygon of the set.

A set of polygons is overlapping if any two complementary subsets have interior points in common.

Two points P_1 and P_2 are said to be mutually accessible with respect to a set of points $[R]$ if there exists a broken line connecting them but, containing no point of $[R]$ except possibly P_1 or P_2 or both.

14. THEOREM. *If two polygons are not identical and have interior points in common, then there are some points of one polygon within the other, and some points of one polygon exterior to the other.*

PROOF. Denote the polygons by p_1 and p_2. Since each polygon is simple, it follows that not all points of either lie on the other. Let P be an interior point of both p_1 and p_2. If there are no points of p_1 within p_2, then all points of p_2 are accessible from P with respect to p_1. Let Q be a point of p_2, not of p_1. Then Q lies within p_1, since it is accessible from the interior point P and does not lie on the polygon itself. In the same manner we show that there are some points of one exterior to the other.

15. THEOREM. *If $[p]$ is a finite set of finite overlapping polygons, there is a finite polygon p' all of whose points are points of $[p]$ such that all interior points of the set $[p]$ are interior points of p'.*

PROOF. On a line l let P be a point such that all intersections of l with $[p]$ lie on the same side of P. Denote by p' all points of $[p]$ accessible from P, and by $[I]$ all points not thus accessible. Then (a) no point of p' lies within a polygon of $[p]$ and every segment of p' is reached from P from the exterior side of the polygon of $[p]$ on which it lies.

(b) All interior points of the set $[p]$ are points of $[I]$, since no such point can be reached from a point exterior to all polygons of $[p]$.

(c) Any two points, both interior, of the set $[p]$ are mutually accessible with respect to p'. Suppose the polygons of $[p]$ are ordered as $p_1, p_2, \ldots, p_n$ in such manner that p_i and p_{i+1} have interior points in common ($i = 1, \ldots, n-1$); then clearly any two interior points of p_i and p_{i+1} are mutually accessible, since one of these polygons contains points which lie within the other (14), and hence are not points of p' (a).

(d) Let I_1 be any point of $[I]$ not an interior point of the set $[p]$. Connect I with a point Q on a segment of a polygon p_1 of $[p]$. Then $I_1 Q$ is exterior to the polygon p_1, while I_1 is not accessible from P with respect to p'. Hence, by (a), Q is not a point of p' whence I_1 can be joined to a point within p_1 without meeting p'. Hence $[I]$ is a connected set with respect to p'. Clearly the set of points not of p' which are accessible from P is a connected set. Hence p' is a finite set of segments separating the remaining points of the plane into two connected sets. Clearly no subset of p' does thus separate the plane, since removing a single point from p' enables us to reach points of $[I]$ from P. Hence, by (27), p. 44, p' is a simple finite polygon.

DEFINITION. *A point L is a "limit-point" of a set of points $[P]$ if there are points of $[P]$ other than L within every triangle of which L is an interior point.*

16. Theorem. *An exterior point of a polygon is not a limit-point of interior points, and an interior point is not a limit-point of exterior points.*

Proof. This is a direct consequence of (4) and (14).

17. Theorem. *A broken line which lies entirely within a polygon, except its end-points which lie on the polygon, forms with the polygon two polygons having no interior point in common such that all interior points of the first polygon lie on or within the two resulting polygons.*

Proof. Denote the broken line by $P_1 P_2$. It is a consequence of the definition of polygon that two polygons are thus formed, the broken line $P_1 P_2$ being part of each polygon. Denote these two polygons by p_1 and p_2. Since every point of each polygon not of $P_1 P_2$ is accessible from some external point, it follows that neither polygon contains a point within the other, and hence by (14) they have no common interior point. That every interior point of the original polygon is on or within p_1 or p_2 is a direct consequence of the definition of interior points.

Definition. *A broken line b is said to cross a polygon p once between two points P_1 and P_2 of b if one of the two points, as P_1, is exterior and the other is interior, and if, following b from P_1 to P_2, one is never led back from interior to exterior points. The polygon is also said to cross the broken line.*

It will be noticed that some segments of b and p may coincide.

18. Theorem. *A broken line AB, finite or at most having the limit-vertices A and B, crosses a polygon p an odd number of times if A is exterior and B is interior and if AB contains no limit-vertex of p.*

Proof. This is an immediate consequence of (13).

19. Theorem. *If p_1 is a finite polygon or infinite at most at the points A and B, and if p_2 is a finite polygon of which A is exterior and B is interior, then p_2 contains a broken line which connects a point on one of the broken lines AB of p_1 with a point of the other broken line AB of p_1, and which lies entirely within p_1.*

Proof. The polygon p_1 contains two broken lines AB which we denote by b_1 and b_2. By (18) each of the broken lines crosses the polygon p_2 on odd number of times. The polygons p_1 and p_2 clearly have interior points in common, since points of the one lie within the other, and hence by (14) there are points of p_2 exterior to p_1. Let Q be any such point. Suppose the theorem not true. Following the polygon p_2 from the point Q we can meet b_2 only after having crossed b_1 an even number of times (zero being an even number), for otherwise just before meeting b_2 we should trace a broken line within p_1 such as we suppose

does not exist. Similarly we can not meet b_1 again until we have first crossed b_2 on even number of times. Continuing in this way, remembering that p_2 is a finite polygon, we show that p_2 crosses b_1 and b_2 an even number of times, or, what is the same thing, b_1 and b_2 each cross p_2 an even number of times, contrary to (18). Hence the broken line specified in the theorem exists.

20. THEOREM. *If two points A and B are connected by any broken line, finite or having at most the limit-vertices A and B, then there is a subset of this broken line, which forms a simple broken line connecting A and B.*

PROOF. Inclose A and B in the small triangles t_1 and t_2 respectively. Let A_1 be a point of the broken line not within either triangle. From A_1 trace the broken line towards B until we meet a point in the line already traced. Then a complete polygon has been traced, which we now omit from the broken line we are seeking. Since there are only a finite number of such polygons on $A_1 B_1$ exterior to the triangle t_2, we finally obtain a simple broken line connecting A_1 with a point within t_2. Since this is true for any triangle of which B is an interior point, we have a simple broken line connecting A_1 and B. In the same manner we obtain a simple broken line connecting A_1 and A, and these together form the broken line required.

DEFINITIONS. *A set of points is bounded if it lies within a polygon.*

A polygon is convex if for any line on which lies one of its sides there are no points of the polygon on one side of the line.

21. THEOREM. *For any polygon p there exists a convex polygon p_1 such that every interior point of p_1 lies within p.*

PROOF. About each limit-vertex L_i of p set a triangle t_i. Then there is a finite set of exposed segments. Connect every pair of end-points of these segments, forming a finite set of segments $[\sigma]$. Let P be any interior point of p. Then there are points of p and hence of $[\sigma]$ on both sides of every line through P. Draw any half-line from P not meeting an end-point of $[\sigma]$. Then on this half-line there is a finite set of points of $[\sigma]$, and hence a last such point which lies on a segment $Q_1 Q_2$. Then on one side of the line $Q_1 Q_2$ there is no end-point and hence no point of $[\sigma]$, for if there were we should have a line meeting only one side of a triangle. Denote the angle $Q_1 P Q_2$ by α_1. From P draw rays through the various end-points of $[\sigma]$ and order the angles thus formed, making a set $[\alpha_i]$. Since there are points of $[\sigma]$ on both sides of the line $P Q_2$, there are such points on that side of this line which is opposite the ray $P Q_1$. Hence there is an angle of α_i, as α_2, of which $P Q_2$ is a side, whose other side is on the

opposite side of PQ_2 from PQ_1, and within which there is no end-point of $[\sigma]$. Within $\angle \alpha_2$ construct a ray from P meeting $[\sigma]$ in a last segment $Q_2 Q_3$. Again on one side of the line $Q_2 Q_3$ there is no point of $[\sigma]$. In this manner we continue until we reach Q_1. Then the polygon $Q_1 Q_2$, $Q_2 Q_3$, $\cdots$, $Q_n Q_1$ has the required properties.

§ 3. *Concerning a Sequence of Sets of Regions which Close down Uniformly on a Closed Set of Points.*

We now consider a plane in which Axioms I–VIII, C of § 1 hold.

DEFINITIONS. *A set of points is "closed" if it contains all its limit-points.*

A set of points is a "connected set" if at least one of any two complementary subsets contains a limit-point of points in the other set.

A "region" consists of an entirely open connected set together with any or all of those of its limit-points which are not points of the set.[*]

It is only in the presence of Axiom C that a "closed" set as defined in the present paragraph differs from one not closed. The definition of "connectedness" given on page 293 may apply to a plane of Axioms I–VIII or to one of Axioms I–VIII and C, while the definition given in this section applies only in case Axiom C is included. However, the latter definition of connectedness applies in cases where the former does not.

* The terms "connected" and "region" have been defined variously. G. Cantor (*Mathematische Annalen*, Vol. XXI, p. 575) defines "connected" as follows, in terms of geometric congruence. A set of points T is "zusammenhängend, wenn für je zwei Punkte t und t' derselben, bei vorgegebener beliebig kleiner Zahl ε immer eine endliche Zahl Punkte t_1, t_2, $\ldots$, t_ν von T auf mehrfache Art vorhanden sind, so dass die Entfernungen $\overline{t\,t_1}$, $\overline{t_1 t_2}$, $\overline{t_2 t_3}$, $\ldots$, $\overline{t_\nu t'}$ sämtlich kleiner sind, als ε."

W. H. Young, in his "The Theory of Sets of Points," p. 204, gives an equivalent definition in non-metrical terms: "A set of points such that, describing a region in any manner round each point and each limiting point of the set as internal points, these regions always generate a single region, is said to be a *connected set* provided it contains more than one point."

It will be noticed that these definitions make many sets connected which it would seem are not naturally so regarded. Thus, according to them the interior and exterior points of a circle or a triangle belong to the same connected set. A segment is connected though any set of isolated points is removed. In general, if from the ordinary continuum in space of any dimensions any set whatever which is nowhere dense is removed, the residue would form a connected set.

Schoenflies (*Mathematische Annalen*, Vol. LVIII, p. 209), following Jordan ("Cours d'Analyse," Vol. II, p. 25), first defines the notion of connectedness for a perfect set, "Eine perfekte Menge T heisst zusammenhängen, wenn sie nicht in Teilmengen zerlegbar ist, deren jede perfekt ist." Also (p. 210), "*Die Komplimentärmenge M einer zusammenhängenden perfekten Menge T heisst zusammenhängend, resp. zusammenhängendes Gebiet, falls je zwei ihrer Punkte durch einen einfachen Weg verbindbar sind, der ihr ganz angehört.*" Schoenflies then remarks, "Diese Definition ist mit der Cantor'schen inhaltlich übereinstimmend," which is obviously not so. The example given above of the interior and exterior of a circle or a triangle, which under the Cantor definition belong to the same connected set, shows this, since under the Schoenflies definition just given these will not

We remark, in connection with this definition of region, that it is supposed to carry with it an implicit reference to the number of dimensions of the space that is considered. Thus if only the points of a line are considered, a segment of the line is a region. In a plane the interior of any polygon is a region, but this set does not form a region if it is considered in a three-dimensional space.

DEFINITION. *An infinite sequence of segments* $\{\sigma_i\}$ *of a line* l *is said to "close down upon a point* P *as a limit-point" if for every segment* σ' *of* l *containing* P *there is a value of* i, $i = k$, *such that every segment* σ_{k+j} $(i = 0, \ldots, \infty)$ *is contained in* σ'. P *is said to be a limit-point of the sequence* $\{\sigma_i\}$.

1. THEOREM. *If a sequence of segments* $\{\sigma_i\}$ *close down upon a point* P *as a limit-point, then there is no other point* $P' \neq P$ *which lies on every segment of* $\{\sigma_i\}$.

PROOF. Consider a segment containing P of which P' is one end-point. Then there is an infinitude of segments of $\{\sigma_i\}$ which lie on this segment and hence do not contain P'.

2. THEOREM. *For every point* P *of a line* l *there is a sequence of segments* $\{\sigma_i\}$ *on the line* l *of which* P *is a limit-point.*

PROOF. In the figure l'' is a half-line proceeding from R in l $(R \neq P)$, l not containing l''. Let S be a point on the same side of l as l'', such that S and P are on opposite sides of l''. Connect S and P, meeting l'' in S'. Q is any point on l'' in the order $R\,S'\,Q$. Connect P and Q by the line l' and let P_1 be any point of l in the order $R P P_1$. From S' project P_1 into P_1' on l', and from S project P_1' into P_2 on l, and so on. Continuing in this manner, using S and S' as centers of projection, we obtain a sequence of points P_1, P_2, P_3, $\ldots\ldots$

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belong to the same connected set. Veblen (*Transactions of the American Mathematical Society*, Vol. VI, p. 91) uses the word "coherence" and defines the same as the Jordan-Schoenflies "Zusammenhang."

The term "region" is usually defined in substance as in the text of this paper, but from a variety of points of view and with varying degrees of complexity of statement. Veblen (*loco citato*, p. 85), however, defines "region" as "a set of points, any two of which are points of at least one broken line composed entirely of points of the set." This definition of "region" makes any broken line a region while an arc of a circle is not. The definition given by Young is (*loco citato*, p. 180): "A part of the plane which can be tiled over by a transitive set of triangles is called a domain or completely open region.".... "The most general form of region consists of a domain together with some or all of its non-included limiting points."

The term "transitive," when applied to a set of triangles, is previously defined as follows: "Given a set of triangles, whose equivalent primitive triangles are d_1, d_2, $\ldots$, it may be that we can find a proper component of this set, d_{i_1}, d_{i_2}, $\ldots$, such that no triangle of this component overlaps with any but triangles of this component. If so, the set is said to be intransitive, otherwise transitive." The equivalent primitive triangles are triangles having rational points as vertices and containing the same interior points.

We now assume as an axiom that P is a limit-point of this sequence.[*]

A similar sequence of points Q_1, Q_2, ... on the segment PR of which P is also a limit-point gives the sequence of segments $\{P_iQ_i\}$ of which P is a limit-point.

3. THEOREM. *If in the figure used in proving* (2) *a point K is added in the order* PKP_1, l'', S, Q *and* P_1 *remaining fixed, then, in the sequences* $\{P_i\}$ *and* $\{K_i\}$ *approaching* P *and* K *respectively* $(P_1=K_1)$, P_i *lies between* P *and* K_i *for* $i \geqq 2$.
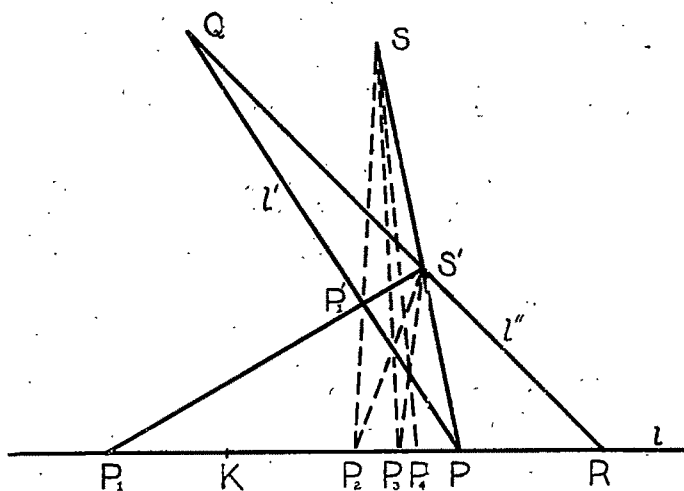
FIG. 4.

PROOF. This follows by mathematical inductions from elementary theorems on the interior and exterior of a triangle.

DEFINITION. *A sequence of sets of regions* $\{[R]_i\}$ *is said to close down uniformly upon a set of points* $[P]$ *if* (a) *every point* $[P]$ *is an interior point of some region of every set* $[R]_i$ *of* $\{[R]_i\}$.

(b) *For every finite set of regions* $[R]'$ *which contains every point of* $[P]$ *as interior points there is a value of* i, $i = k$, *such that every region of every set* $[R]_{k+j}$ $(j = 0, ..., \infty)$ *lies entirely within some region of* $[R]'$.

. * Von Staudt ("Geometrie der Lage," p. 50) uses essentially this construction in proving the fundamental theorem of projective geometry, but makes use of no axiom such as in the text. Klein (*Mathematische Annalen*, Vol. VI, p. 140) pointed out that the argument of Von Staudt is not conclusive. Klein uses a stronger axiom than the one here used; viz., that a limit-point (finite) of any sequence (bounded) exists. The axiom in its weaker form here used corresponds for projective geometry to the Archimedean axiom for metric geometry; viz., that for any two fixed segments $A_1 A_2$ and σ one can apply σ to $A_1 A_2$ a finite number of times and thus completely cover it. The theorem may of course be proved without the use of this special axiom if we assume the full axiom of continuity, p. 291.

4. THEOREM. *For every closed bounded set of points $[P]$ there is a sequence of finite sets of regions $\{[R]_i\}$ which closes down uniformly upon the set $[P]$.*

PROOF. (a) We consider first the case when the set is contained in a segment AB of a line l. Select the points P_1 and R in the order $P_1 A B R$. Construct as in the proof of (2) a sequence of segments $\{A_i' B_i'\}_P$ for every point P of the segment AB, using the same points Q, S, R and P_1 for all points P of $[P]$. Then each set $[A_i' B_i']$ (i fixed) consists of an infinite set of segments of which every point of $[P]$ is an interior point. Since $[P]$ is a closed set, it follows by the Heine-Borel theorem* that there is a finite subset of $[A_i' B_i']$, $[AB]_i$, within which lie all points of $[P]$. We now show that $\{[AB]_i\}$ ($i = 1, \ldots, \infty$) is the required sequence of sets of regions. Let $[\sigma_i]$ be any finite set of segments such that every point of $[P]$ lies within at least one segment of the set. Consider any such segment σ_i whose end-points are $C_i D_i$. Then, since $[P]$ is a closed set, either C_i lies within a segment of $[\sigma_i]$ or there is a point C_i' of $C_i D_i$ such that $C_i C_i'$ contains no point of $[P]$. In case C_i lies on a segment of $[\sigma_i]$, a point C_i' is chosen on $C_i D_i$ so that $C_i C_i'$ lies entirely within this segment. Points D_i' are chosen in a similar manner with respect to D_i. The segment $C_i' D_i'$ having been thus constructed for a particular value of i, care is taken in constructing these segments for the other values of i so that every point of $[P]$ shall lie within $[C_i' D_i']$. It is understood that all points C_i, D_i, C_i' and D_i' are in the order $P_1 C_i D_i R$, and $P_1 C_i' D_i' R$.

Consider now any particular segment of $[\sigma_i]$, as $C_1 D_1$. In the sequence of points $\{P_n\}_{C_1}$ approaching C_1' there is by the definition of limit-point a value of n, n_1 such that all points $\{P_{n_1+j}\}_{C_1'}$ ($j = 0, \ldots, \infty$) lie on $C_1 C_1'$. But by (3) $\{P_{n_1+j}\}_C$, where C is any point of $C_1' C_2'$, lie on $C_1 D_1$. We thus obtain such value of n, n_i for every segment of $C_i D_i$. Let N' be the largest of the finite set of numbers n_i. Then all points $[\{P_{N'+j}\}]$ ($j = 0, \ldots, \infty$) lie on a segment $C_i D_i$. Similarly we obtain the points $[\{P_{N''+j}\}]$ approaching the points of $[P]$ from the side on which R lies. If N is the greater of N' and N'', then in the sequence $\{[AB]_i\}$, described above, every set $[AB]_i$ for $i \gtrless N$ lies within $[\sigma_i]$.

(b) Next let $[P]$ be any closed bounded plane set. By (21), §2, there is a

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convex polygon p within which lie all points of $[P]$. It follows at once from (a) that there is a sequence of finite sets of segments which closes down uniformly on the set of all points of the polygon. Denote this sequence by $\{[AB]_i\}$. (We here include in "segment" the simple case of a broken line consisting of two segments whose common end-point is a vertex of the polygon.) Let P_1 and P_2 be two vertices of the polygon and the segment $P_1 P_2$ one of its sides. Connect each of the points P_1 and P_2 with the extremities of each segment of $[AB]_i$ for all values of i. Thus for each value of i we obtain the set of polygonal regions into which these segments separate the region enclosed by the polygon p. Denote by $[R]_i$ a finite subset of this set of regions such that there are points of $[P]$ within every region of $[R]_i$. We now prove that $\{[R]_i\}$ is a sequence of sets of regions of the type specified in the theorem.
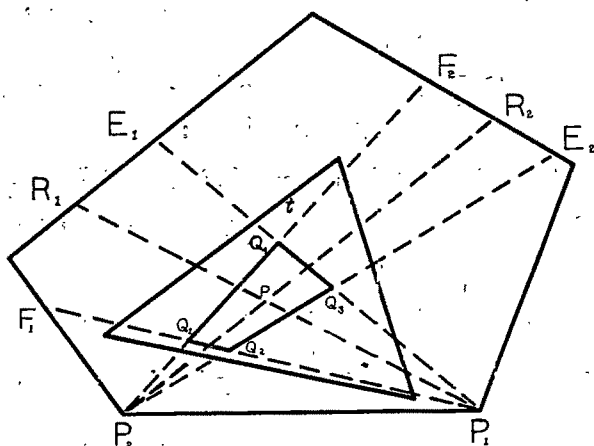
Fig. 5.

Let $[R]$ be any finite set of regions of which all points of $[P]$ are interior points. About every point P of $[P]$ construct a triangle t lying within a region of $[R]$. Through P_1, P_2 and P construct segments $P_1 R_1$ and $P_2 R_2$ as shown in the figure. By means of these segments we can now construct $P_1 E_1$, $P_1 F_1$, $P_2 E_2$, $P_2 F_2$, such that the quadrilateral $Q_1 Q_2 Q_3 Q_4$ formed by them shall contain P as an interior point and lie within the triangle t. Then the regions of the set consisting of all regions of the type $Q_1 Q_2 Q_3 Q_4$ lie within regions of $[R]$ and contain all points of $[P]$ as interior points. By the Heine-Borel theorem there is a finite subset $[R]'$ of this set of regions which fulfils these conditions. Consider now the set of segments $[EF]$ consisting of all segments of the polygon p except the segments $P_1 P_2$ into which the points E_1, F_1, E_2, F_2, etc., separate it.

Then there is a value of i, $i = i_1$, such that, in the sequence of sets of segments $\{[AB]_i\}$, every segment of every set of $\{[AB]_i\}$ lies within a segment of $[EF]$ for every value of i such that $i \leqq i_1$. Hence in the sequence of regions $\{[R]_i\}$ every region of every set for $i \leqq i_1$ lies within a region of the set $[R]'$, and hence within a region of $[R]$. Hence $\{[R]_i\}$ has the required properties.

§ 4. *Definition of Continuous Simple Curve.*

DEFINITION. *If every point of each of the sets $[P]'$ and $[P]''$ is a point of a set $[P]$, then we say that $[P]$ is the sum of the two sets $[P]'$ and $[P]''$ and write $[P]' + [P]'' = [P]$. This does not imply that the sets $[P]'$ and $[P]''$ have no elements in common.*

1. THEOREM. *If each of the sets of points $[P]'$ and $[P]''$ is a connected set and has at least one point in common with the other, then the set $[P] = [P]' + [P]''$ is a connected set.*

PROOF. Let $[\overline{P}]$ and $[\overline{\overline{P}}]$ be any pair of complementary subsets of $[P]$. Then one of the following statements must hold:

(a) $[\overline{P}] \equiv [P]'$ or $[\overline{P}] \equiv [P]''$ and $[\overline{\overline{P}}] \equiv [P]''$ or $[\overline{\overline{P}}] \equiv [P]'$.

(b) There are points of at least one of the sets $[P]'$ and $[P]''$ in both $[\overline{P}]$ and $[\overline{\overline{P}}]$.

In case (a) $[\overline{P}]$ and $[\overline{\overline{P}}]$ have at least one point in common, whence either set contains a limit-point of points of the other set.

In case (b) it follows from the connected character of $[P]'$ and $[P]''$ that one of the sets $[\overline{P}]$ and $[\overline{\overline{P}}]$ contains a limit-point of points in the other set, whence the theorem is proved.

DEFINITION. *A continuous simple arc connecting two points A and B, $A \neq B$, is a bounded, closed, connected set of points $[A]$ containing A and B such that no connected proper subset of $[A]$ contains A and B.*

We speak of this arc as the arc AB or BA, A and B being called the endpoints of the arc. We note that a line-interval is an arc according to this definition.

2. THEOREM. *Every point A_0 of an arc AB, distinct from both A and B, separates in a unique way the remaining points of the arc into two sets, one containing A and the other containing B, such that the set containing A, together with A_0, forms an arc AA_0 and the set containing B, together with A_0, forms an arc BA_0. The arcs AA_0 and BA_0 have no point in common except A_0.*

PROOF. (a) By the definition of arc the points of the arc AB apart from A_0 form at least one pair of complementary subsets, one set containing A and the other containing B, such that neither set contains a limit-point of the other. Consider one* pair of such sets. Adjoin A_0 to each set and denote the set containing A by AA_0 and the set containing B by BA_0. We also denote the set forming the arc AB by $[A]$.

(b) *The sets AA_0 and BA_0 are closed.*

By hypothesis all limit-points of AA_0 are points of $[A]$, $[A]$ being closed; and since BA_0 contains no limit-point of points of AA_0 (except possibly A_0), it follows that all such limit-points must be points of AA_0; that is, AA_0 forms a closed set. Similarly BA_0 is a closed set.

(c) *Each of the sets AA_0 and BA_0 is connected.*

Suppose that one of these sets, as AA_0, is not connected; *i. e.*, contains two non-vacuous complementary subsets neither one of which contains a limit-point of points of the other. To that one of these sets which contains A_0 add the set BA_0. Then we should have a pair of non-complementary subsets of $[A]$ neither of which contains a limit-point of the other, so that $[A]$ would not be a connected set.

(d) *The set $\begin{bmatrix} AA_0 \\ BA_0 \end{bmatrix}$ does not contain a connected proper subset containing $\begin{bmatrix} A \\ B \end{bmatrix}$ and A_0.*

If AA_0 contains a proper connected subset $\overline{AA_0}$ containing A and A_0, then, by (1), $\overline{AA_0} + BA_0$ form a connected set containing A and B, which is contrary to the definition of arc.

It follows from (a)-(d) that AA_0 and BA_0 are arcs. We refer to them as complementary arcs a and b of AB.

(e) *The set $[A]$ contains only one pair of complementary arcs connecting A and A_0 and B and A_0.*

Suppose there are two such pairs of arcs, a, b and a', b'. Since a and b contain together all points of $[A]$, it follows that a and b contain all points of a'. If not all points of a are in a', then the subset of a' which is in a is not connected. But adding any set of points from b to this subset of a must fail to make it connected, since neither of the sets a and b contains a limit-point of the other except A_0. Hence a' would fail to be connected. In the same manner we show

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* Under (e) we show that there is only one such pair.

that all points of a' are points of a, whence a and a' are identical. In the same manner b and b' are identical.

DEFINITION. *A point " on an arc " is any point of the arc, including the end-points. A point " within an arc " is any point of the arc not an end-point.*

3. THEOREM. *If A_0 is a point within the arc AB, and A_1 any point within the arc AA_0, then the arc A_1B contains every point of A_0B.*

PROOF. The arc AA_1 contains no point of A_0B, since $AA_1 + A_1A_0 = AA_0$ contains no point of A_0B except A_0. Hence A_1B contains all points of A_0B.

4. THEOREM. *Any two points of an arc determine uniquely an arc connecting them.*

DEFINITION. *Any point A_1 within an arc AB is said to lie between the points A and B on the arc. We also say that the points A, A_1, B are in the order AA_1B or BA_1A on the arc AB.*

5. THEOREM. *For any four points on an arc a notation may be so arranged that we shall have the orders ABC, ABD, ACD, BCD.*

PROOF. This is an immediate consequence of (4) and (3).

6. THEOREM. *If A is an interior point of a polygon and B an exterior point, then every arc AB contains a point of the polygon.*

PROOF. Suppose there are two complementary subsets of the arc AB such that one lies outside the polygon and the other inside the polygon; then, by (16), § 2, neither of these sets contains a limit-point of the other and hence the arc AB would not be connected.

7. THEOREM (Ordinal Continuity of an Arc). (a) *If A_1 and A_2 are any two points of an arc, then there is a point A_3 on the arc in the order $A_1A_3A_2$.*

(b) *If $[A]'$ and $[A]''$ are complementary subsets of the points of an arc AB such that no point in either set lies between points of the other set on the arc, then aside from A and B there is one and only one point of the arc which does not lie between points of either set.*

PROOF. (a) is a direct consequence of (4) and the *connected* property of an arc.

(b) Let $[A]'$ and $[A]''$ be any pair of complementary subsets of an arc AB such that no pair of either lies between points of the other. Then there is a point A_0 in one of these sets, as $[A]'$, which is a limit-point of points in the other set. The points of $[A]''$ lie entirely on one of the arcs AA_0 and BA_0, as BA_0, for otherwise we should have the point A_0 of $[A]'$ between points of $[A]''$. Suppose now there is a point A_1 of $[A]'$ on the arc BA_0; then, since A_0 is a limit-point of $[A]''$, there are points of $[A]''$ between A_1 and A_0, which are both of $[A]'$. Hence there is no point of $[A]'$ on BA_0, and A_0 is therefore the required point.

8. THEOREM (Geometric Continuity of an Arc). *If A_0 is any point of an arc AB, and t_1 any triangle containing A_0 as an interior point, then (in case $A_0 \neq A$) there is a point A_1 on the arc AA_0 and (in case $A_0 \neq B$) a similar point B_1 on the arc BA_0 such that every point of the arc $A_1 B_1$ lies within t_1.*

PROOF. Let t_2 be any triangle within t_1 also containing A_0 as an interior point. We consider the arc AA_0. Suppose A exterior to t_2. Then by (6) there are points of AA_0 on t_2. If the theorem does not hold, then, for any point $A_2^{(1)}$ of AA_0 on t_2, there are by (6) points of $A_2^{(1)}A_0$ on t_1. Let $A_1^{(1)}$ be such a point. Further, there is a point $A_2^{(2)}$ of the arc $A_1^{(1)}A_0$ on t_2, a point $A_1^{(2)}$ of the arc $A_2^{(2)}A_0$ on t_1, etc. In this manner we obtain an infinite sequence of points $\{A_1^{(i)}\}$ of AA_0 on t_1 and a sequence $\{A_2^{(i)}\}$ of AA_0 on t_2. Let A_1' be a limit-point of $\{A_1^{(i)}\}$ and A_2' a limit-point of $\{A_2^{(i)}\}$.*

The points A_1' and A_2' can not lie on an arc $A_1^{(i)}A_2^{(i)}$, for in that case such arc would contain a limit-point of the arc $A_1^{(i+1)}A_0$, which is impossible. Further, A_1' and A_2' are points of the arc AA_0. Suppose we have the order $A_1'A_2'A_0$. Then all the arcs $A_2^{(i)}A_2^{(i+1)}$ lie on the arc AA_1'; for suppose one such arc, as $A_2^kA_2^{(k+1)}$, lies on $A_1'A_0$, then all subsequent arcs of the sequence lie on this arc, and hence A_1' can not be a limit-point of points of these arcs. But if all arcs of the sequence $A_2^{(i)}A_2^{(i+1)}$ lie on AA_1', then A_2' can not be a limit-point of these arcs. Similarly for the order $A_2'A_1'A_0$.

9. THEOREM. *If A_0 is any point of an arc AB, and t_1 any triangle containing A_0 as an interior point, then there exists a triangle t_2 containing A_0 as an interior point such that every arc of AB which connects A_0 with a point of t_2 lies entirely within t_1.*

PROOF. Let A_1 be a point on the arc AA_0 such that no point of the arc A_1A_0 is on or exterior to the triangle t_1, (8), and B_1 a similar point on A_0B. Then A_0 is not a limit-point of points on the arcs AA_1 and BB_1. Hence, by definition of limit-point there is a triangle t_2 within t_1, and containing A_0 as an interior point, such that there is no point of AA_1 and BB_1 on or within t_2. Hence t_2 is the required triangle.

10. THEOREM. *The points of any two arcs may be set into complete one-to-one correspondence preserving order.†*

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* The existence of the points A'_1 and A'_2 follows from axiom C by well-known argumentation.

† Professor Veblen has proved ("Theory of Plane Curves in Non-Metrical Analysis Situs," *Transactions of the American Mathematical Society*, Vol. VI (1905), pp. 83–98) that two sets of points possessing the order relations specified under (7) and (8) may be thus set into a one-to-one correspondence 'Veblen's proof consists in showing that any set having these properties contains a numerably infinite set of points which is everywhere dense in the set, and then applying a theorem of G. Cantor (*Mathematische Annalen*, Vol. XLVI (1895), pp. 481– 512) to the effect that all sets having this property together with those given by the theorems (2), (7), (8) may be thus set into correspondence. However, Veblen's proof involves metric relations, inasmuch as he makes use of equal segments.

PROOF. Let $\{[t]_i\}$ be an infinite sequence of finite sets of triangular regions closing down uniformly on the points of an arc AB (see § 3). Let $[A]_t$ be a finite set of points of AB containing at least one point within each triangle of the set $[t]_i$ such that $[A]_t$ contains all points of $[A]_{i-1}$ for all values of i. Then the infinite sequence $\{[A]_i\}$ contains a numerably infinite set of points which is everywhere dense on the arc AB. The theorem now follows from the theorem of G. Cantor cited in the foot-note. The proof may also be completed very easily as follows. For any two arcs AB and $A'B'$ the sets $\{[A]_i\}$ and $\{[A']_i\}$ may obviously be set into correspondence preserving order. In order that $[A]_i$ and $[A']_t$ shall contain the same number of points we add the requisite number of points to one of them. Let A_0 be any point of AB not of $\{[A]_i\}$. Then A_0 separates $\{[A]_i\}$ into two sets neither one of which contains a point between points of the other. There will then be a corresponding division of $\{[A']_i\}$, whence, by (7), there is a point A'_0 which we now set in correspondence with A_0. In this manner all points of the two arcs are set into a one-to-one correspondence. That order is preserved is obvious.

§ 5. The Frontier of a Region.

DEFINITIONS. *Consider an entirely open bounded region R enclosed in a polygon p such that there is no limit-point of R on p. Denote by $[E']$ all points of the plane accessible from p with respect to R, and by $[F]$ all common limit-points of $[E']$ and R. Denote by $[I]$ all points of the plane which are contained in neither of the sets $[E']$ and $[F]$, and by $[E]$ all points of $[E']$ not of $[F]$.*

$[F]$ is called the " frontier " of the set $[I]$. $[I]$ is the interior set of $[F]$, and $[E]$ its exterior set.

A point F_1 of $[F]$ is said to possess exterior accessibility if there exists a finite or continuous infinite broken line connecting it with a point of $[E]$, and to possess internal accessibility if there exists a finite or continuous infinite broken line connecting it with a point of I, the broken line in either case containing no point of $[F]$ except F_1.

1. THEOREM. *If every point of a frontier $[F]$ possesses external accessibility, it separates the remaining points of the plane into two connected sets $[E]$ and $[I]$.*

PROOF. (a) By definition any broken line connecting a point of $[E]$ with a point of $[I]$ meets $[F]$, for otherwise some point of $[I]$ would be accessible from a point of the bounding polygon p.

(b) Any two exterior points are mutually accessible, since all such points are accessible from points of p.

(c) Any two interior points I_1 and I_2 are mutually accessible with respect to $[F]$ if both lie in R. This is an immediate consequence of the entirely open, connected and bounded character of R.

(d) Any two interior points I_1 and I_2 are mutually accessible with respect to $[F]$. If one of these points, as I_1, is not a point of R, we need only to prove that I_1 is accessible from some point of R with respect to $[F]$. Join I_1 to F_1 and F_2 of $[F]$ by means of segments $I_1 F_1$ and $I_1 F_2$, neither of which contains a point of $[F]$. That such segments exist is an immediate consequence of axiom C and the closed character of $[F]$. Connect F_1 and F_2 with points of p by means of continuous simple broken lines, (20), § 2, containing no points of $[F]$ except F_1 and F_2. By (17), § 2, these broken lines, together with the polygon p, form two polygons having no interior points in common. There are points of $[F]$ and hence, by (16), § 2, points of R within each polygon, for otherwise I_1 would be accessible from p. Since R is a connected set, there must be points of R on each polygon. But all segments of these polygons except $I_1 F_1$ and $I_1 F_2$ lie in $[E]$. Hence there are points of R on one of the segments $I_1 F_1$ and $I_1 F_2$ whence I_1 is accessible from some point of R.[*]

2. THEOREM. *If every point of a frontier $[F]$ possesses both interior and exterior accessibility, then any two points F_1 and F_2 of $[F]$ may be connected by two simple broken lines, one in $[I]$ and one in $[E]$, and these two broken lines form a polygon which separates the remaining points of $[F]$ into two sets each of which is a continuous arc connecting F_1 and F_2.*

PROOF. The existence of such broken lines is an immediate consequence of the twofold accessibility of every point of $[F]$ and the connected character of $[E]$ and $[I]$ and (20), § 2. By definition these broken lines form a polygon p'. Let E_1 be an exterior point of $[F]$ on p and I_1 an interior point of $[F]$ on p'. Then there are points of $[F]$ both exterior and interior to p', for otherwise I_1 and E_1 would be mutually accessible with respect to $[F]$, (12), § 2. Denote by $[F]'$ the points of $[F]$ within p', together with the points F_1 and F_2.

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[*] We note that (a), (b), (c) follow from the definition of frontier without the use of the special assumption of exterior accessibility. That (d) does not follow without this special assumption is shown by the following example: Consider a circle with two spirals, each going around the circle an infinite number of times and approaching it but having no point in common. Connect these spirals by means of a segment. Then the spirals, together with the segment, enclose a region R which contains no interior point of the circle. The set $[I]$ defined by this region contains also the interior of the circle and is thus not connected.

(a) Since $[F]$ is a closed set, it follows from (16), § 2, that $[F]'$ is closed.

(b) There is no connected proper subset of $[F]'$ containing F_1 and F_2; for if there is such subset, let F' be a point of $[F]'$ but not of the connected subset. Then F' may be connected with E_1 and I_1 by means of a broken line connecting no other point of $[F]$. It is evident that these broken lines may be so chosen as to lie entirely within p'. Then two polygons are formed such that the points of $[F]'$, except F_1, F_2 and F', lie within one or the other of the polygons. Since $F_1 \neq F_2$ it follows that there are points of $[F]'$ within each polygon, and by (16), § 2, these do not form one connected set unless F' is included.

(c) $[F]'$ is a connected set. Suppose $[F]'$ is not connected and that $[F]'_1$ is any closed subset of it which contains no limit-point of the complementary set $[F]'_2$. Suppose A is a point of $[F]'_1$. About A set a triangle containing no point of $[F]'_2$. About every other point of $[F]'_1$ set a triangle lying within p' and on or within which lie no points of $[F]'_2$. Then, by the Heine-Borel Theorem, there is a finite subset of these triangles within which lie all points of $[F]'_1$. By (15), § 2, there is a finite polygon p'' which incloses this set of points, but which does not contain the point F_2. Hence, by (19), § 2, there is a broken line of p'' lying within p', connecting a point on the exterior broken line of p with a point of the interior broken line of p' and not meeting $[F]$. But this contradicts (1).*

DEFINITION. *The set of points consisting of two continuous arcs, each connecting a pair of distinct points A and B and having no other point in common, is called a simple closed Jordan curve, or simply a Jordan curve. We denote it by j.*

3. THEOREM. *If every point of a frontier $[F]$ possesses both internal and external accessibility, it is a Jordan curve.*

PROOF. This is an immediate consequence of the definition of Jordan curve and (2).

§ 6. *Separation of the Plane by a Jordan Curve.*

In § 4 we showed that the points of a continuous arc, as there defined, may be set into a one-to-one correspondence with the points of a straight line interval preserving order. In § 5 a proof is given that the frontier of a region accessible

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* It does not follow that the points of a frontier possess internal accessibility even if they possess external accessibility. This is shown by the following well-known example: The point $\left(\frac{2}{\pi}, 1\right)$ on the curve $y = \sin \frac{1}{x}$ is connected to the point $(0, 1)$ by means of a broken line containing no point of the curve.

from both exterior and interior points is a Jordan curve; that is, a curve consisting of two non-intersecting arcs connecting the same two points. In the present section a proof is given of the converse theorem; viz., that a Jordan curve separates the remaining points of the plane into two entirely open sets.*

For the purpose of studying a Jordan curve, denoted by j, we construct a polygon p having the following properties: Two points P_1 and P_2 of j are on p, and all the remaining points of j are interior points of p. To construct such a polygon let p' be any convex polygon, (21), § 2, within which lie all points of j. Since j is a closed set of points, we obtain by the axiom of continuity an angle with its vertex A_1 one of the vertices of p', such that there are points of j on each side of the angle but no points of j exterior to the angle. Let P_1 and P_2 be points of j, one on each side of the angle. Connect P_1 with the polygon p' by means of two segments exterior to the angle, and similarly for P_2. Then these four segments, together with a properly chosen subset of p', form the required polygon p.

The points P_1 and P_2 separate the polygon p into two broken lines which we denote by b_1 and b_2, and the curve j into two arcs which we denote by a_1 and a_2. The following propositions are stated in terms of this notation.

1. THEOREM. *If an arc a_1 with end-points P_1 and P_2 on a polygon p lies entirely within p, except P_1 and P_2, then some but not all interior points of p are accessible from b_1 with respect to a_1 by means of broken lines lying within*

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* This classic theorem was first stated and proved by Jordan (C. Jordan, "Cours d'Analyse," Paris, 1893, 2d ed., p. 92). For a brief characterization of the literature on this subject, see O. Veblen, *Transactions of the American Mathematical Society*, Vol. VI, pp. 83–98.

The definition given by Jordan in terms of analytic geometry is as follows:

Consider two equations

$$\begin{cases} x = f(t), \\ y = \varphi(t), \end{cases}$$

where $f(t)$ and $\varphi(t)$ have the following properties:

(a) t takes all values of an interval $a \ldots b$.

(b) $f(t)$ and $\phi(t)$ are continuous functions of t on the interval $a \ldots b$.

(c) $f(a) = f(b)$ and $\phi(a) = \phi(b)$.

(d) There is no pair of distinct values of t, t_1 and t_2 such that $f(t_1) = f(t_2)$ and $\phi(t_1) = \phi(t_2)$.

The curve defined by these equations we may now readily show is identical with the Jordan curve defined in § 5.

Let P_1' and P_2' be any two points of $a \ldots b$, $P_1' \neq P_2'$, not both being end-points of the interval. These points separate the interval into two sets, one set consisting of the points lying between the two points, and the other set consisting of the remaining points of the interval. It is clear that the points P_1 and P_2 of the curve corresponding to P_1' and P_2' of the interval separate the curve into two parts and that each part is a continuous simple arc; viz., each is a closed, bounded, connected set containing P_1 and P_2 which has no connected proper subset connecting these points.

p. No point of b_2 is thus accessible from b_1, and any point accessible from b_1 is not accessible from b_2.

PROOF. (a) Since no points of a_1, except P_1 and P_2, are limit-points of b_1, it follows that there are interior points of p accessible as stated in the theorem.

(b) If every interior point of p is so accessible, it follows that points of b_2 are accessible, since the points of b_2 are not limit-points of a_1. But if a point of b_2 is thus accessible, we shall have two polygons with no interior point in common, (17), §2, within each of which lie points of a_1. Since there are no points of a_1 except P_1 and P_2 on these polygons, it follows, (16), §2, that a_1 is not a connected set of points, which is contrary to the definition of a_1. If the same point not on a_1 were accessible from both b_1 and b_2, then a point on b_2 would be accessible from b_1.

Denote the set of points within p and not of a_1 which are accessible from b_1 by $[S]_1$, and the remaining points within p and not of a_1 by $[S]_2$.

2. THEOREM. If any point of the arc a_2 lies in one of the sets $[S]_1$ and $[S]_2$ into which a_1 separates the remaining interior points of p, then all points of a_2 lie within this set; and hence if a point of a_1 is accessible from b_1, no point of a_2 is accessible from b_1.

PROOF. The theorem follows from the connected character of a_2 when we establish that neither of the sets $[S]_1$ and $[S]_2$ has a limit-point of the other. If Q, a point of $[S]_1$, is a limit-point of points in $[S]_2$, we can construct a triangle containing Q as an interior point but no point of a_1, a_1 being a closed set and hence Q not a limit-point of a_1. But there are points of $[S]_2$ within this triangle, and hence Q is accessible from b_2, which is contrary to the definition of $[S]_1$ and $[S]_2$.

3. THEOREM. There is a set of points, not on j, which is not accessible from p by means of any broken line whatever which contains no point of j.

PROOF. Connect a point H on b_1 with a point K on b_2 by means of a broken line lying within p. Let a_1 be the arc accessible from b_1. Since a_1 is closed, it follows that there is a point P_1, on a_1 and the broken line HK, such that there is no point of a_1 on the broken line HK between P_1 and K. There are points of a_2 between P_1 and K, for otherwise the point P_1 on a_1 would be accessible from b_2. Since a_2 is closed, there is a point P_2 on a_2 and the broken line HK, such that there is no point of either a_1 or a_2 between P_1 and P_2. Since P_1 is not accessible from b_2 and P_2 is not accessible from b_1, it follows that the points of the broken line $P_1 P_2$ are not accessible from either b_1 or b_2. Hence these points are of the required type.

DEFINITION. *Every point of the plane, not of j, which is accessible from points of p by any broken line whatsoever, is an exterior point of j; a point not so accessible is an interior point.*

4. THEOREM. *The exterior and interior character of a point with respect to a given curve, as here defined, is independent of the polygon p.*

PROOF. Consider any two polygons p' and p'' such that no point of the curve j is exterior to either of them. Since any point on either polygon can be connected with any point on the other by a broken line containing no point of j, it follows that any point of the plane which can be connected with a point of one of these polygons can be connected with the other.

5. THEOREM. *Every point of j is accessible from p with respect to j.*

PROOF. Let A be any point of a_1. Consider a sequence of triangles t^i closing down upon A as a limit-point, (4), § 3, and having the following properties:

(a) Every triangle lies within the polygon p.

(b) Every point of the arc $P_1 A$ which lies between two points of t_i is an interior point of t_{i-1} $(i = 2, \ldots, \infty)$, (9), § 4.

(c) t_i lies within t_{i-1} $(i = 2, \ldots, \infty)$.

Let A_i be a set of points on the arc $P_1 A$ such that A_i lies on the triangle t_i. About every point of $P_2 A$ except A construct a triangle t, forming a set $[t]$ having the following properties:

(a) No point of the arc $A P_2$ is on or within one of the triangles.

(b) Those triangles of $[t]$ which are constructed about the points of the arc $A_{i-1} A_i$ are interior to t_{i-2} and exterior to t_{i+1}.

The triangles containing as interior points the points of the arc $A_{i-1} A_i$ contain, according to the Heine-Borel Theorem, a finite subset of triangles such that every point of $A_{i-1} A_i$ is an interior point of the set. Since the arc $A_{i-1} A_i$ is a connected set, the set of triangles is overlapping, whence, by (15), § 2, there is a finite polygon within t_{i-2} and exterior to t_{i+1} within which lie all points of $A_{i-1} A_i$. That no point of the arc $A_{i+1} P_2$ lies within this polygon follows from the connected character of the arc and the two facts that the polygon contains no point of $A_{i+1} P_2$ and that P_2 is surely exterior to the polygon.

We thus obtain an infinite sequence $\{p_i\}$ of finite polygons having the following properties:

(a) p_i contains all points of $P_{i-1} P_i$ as interior points $(i = 2, \ldots, \infty)$.

(b) p_i lies within t_{i-2} and exterior to t_{i+2} $(i = 3, \ldots, \infty)$. (p_1 contains the arc $P_1 A_1$ and is exterior to t_2.)

Then by (15), § 2, a certain subset of p_1 and p_2 forms a polygon $p^{(1)}$ which contains all points of $P_1 A_2$ as interior points. Similarly a certain subset of $p^{(1)}$ and p_3 forms a polygon $p^{(2)}$ which contains all points of $P_1 A_8$ as interior points; and, in general, a certain subset of $p^{(i)}$ and p_{i+2} forms a polygon $p^{(i+1)}$ which contains all points of $P_1 A_{i+2}$ as interior points, and within which lie no points of the arc $A_{i+3} P_2$. Also every segment of $p^{(i+1)}$ which is exterior to t_i is a segment of $p^{(i)}$. Further, there is no point of a_1 on $p^{(i+1)}$ except within the triangle t_i. Tracing the polygon $p^{(i+1)}$ from a point on the polygon p, let Q_i be the first point reached on p_i. Then we obtain a sequence $\{Q_i Q_{i+1}\}$ of finite broken lines forming an infinite broken line such that there are only a finite number of its segments exterior to any triangle of the sequence $\{t_i\}$, while there are segments of the sequence within every such triangle. Hence A is accessible from points on both b_1 and b_2 by means of two distinct broken lines. We now observe that one of these broken lines lies entirely in $[S]_1$ and the other in $[S]_2$. If then a_2 lies in $[S]_2$, A is accessible by means of the broken line lying in $[S]_1$.

6. THEOREM. *There is an interior point of j from which all its points are accessible with respect to j.*

PROOF. By (3) there exists an interior point I of j, and hence an interior segment with its end-points Q_1 and Q_2, one on a_1 and the other on a_2. Connect points on b_1 and b_2 with Q_1 and Q_2, respectively, by means of broken lines containing no points of j except Q_1 and Q_2. Denote by a_1' and a_2' the arcs into which Q_1 and Q_2 separate j. Then, by (17), § 2, we have two polygons p and p one of which contains the arc a_1' and the other the arc a_2'. Then, by (5), every point of a_1' and also of a_2' is accessible from I.

7. THEOREM. *A Jordan curve separates all the remaining points of the plane into two connected sets.*

PROOF. The uniqueness of the exterior and interior sets as defined on p. 317 is established in (4). That no two points, one interior and the other exterior, are mutually accessible with respect to j is a direct consequence of the definition of these sets. That any two exterior points are mutually accessible follows from the fact that both are accessible from points on p.

Let I_1 and I_2 be any two interior points. Through I_1 pass a broken line meeting j in only two points, as in the proof of (6), and producing them to reach p. Then we have two polygons p_1' and p_2' within each of which lie of points of j. Connect I_2 with a point of that polygon within which I_2 does not lie, say p_1'. This connecting broken line must meet p_1' in an interior point whence I_1 on this broken line is accessible from I_2, which completes the proof of the theorem.

§ 7. Concerning a Set of Simple Continuous Arcs Having a Simple Continuous Arc as a Limit.

Denote by R a closed bounded set of points in which two points A and B are connected by an infinite set $[a]$ of simple continuous arcs, A and B being the end-points of each arc of $[a]$.*

The arcs of the set $[a]$ are assumed to satisfy the following condition, which we call *uniform continuity of the arcs with respect to the set.*

If P is any point of R and t_1 is any triangle containing P as an interior point, there exists a triangle t_2 within t_1 containing P as an interior point, such that no arc of $[a]$ contains a point of t_1 between points of t_2.†

Let $[A]$ denote the set of points consisting of all points of the arcs of $[a]$, together with their limit-points. Let $\{[t]_i\}$ be a sequence of sets of triangles enclosing triangular regions which close down uniformly upon the set $[A]$. Denote generically by m_i the sum of the number of triangles of the sets $[t]_1$, $[t]_2, \ldots, [t]_i$ of $\{[t]_i\}$. Then the number of different triangles of $[t]_i$ within which lie points of any arc of $[a]$ is less than m_i.

We now proceed to describe a set of points on a certain subset of the arcs $[a]$ which bear a special relation to the set of triangles $[t]_i$ for a definite fixed value of i. On each arc of $[a]$ select m_i points forming a set $P_{[a]_i, j}$ having the following properties:

(a) On each arc at least one point lies within every triangle of $[t]_i$ within which that arc contains points.

(b) On each arc the order of the points is indicated by the subscript j; viz., the points are in the order

$$A = P_{[a]_i, 1}; \quad P_{[a]_i, 2}; \ldots; \quad P_{[a]_i, m_i-1}; \quad P_{[a]_i, m} = B.$$

For a fixed value of j there is then an infinitude of points of $P_{[a]_i, j}$, one on each arc of $[a]$ which has one or more limit-points. Consider this set for $j = 2$. Let $A_{i, 2}$ be a limit-point of the set $P_{[a]_i, 2}$, and let $P'_{[a]_i, 2}$ be a subset of $P_{[a]_i, 2}$ such

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* Obviously there are closed and connected sets which contain no continuous arcs connecting certain pairs of their points. This discussion does not apply to such sets and such pairs of points. In case there is only one arc in R connecting A and B, or in case all such arcs partly coincide, then the arcs of $[a]$ coincide wholly or in part.

† This condition is the non-metrical equivalent of a condition stated by G. Ascoli in the usual metric terms [G. Ascoli: "Accademia die Lincei," (1884)]. The theorem of Ascoli, which seems to have been neglected, is capable of extension and important applications. This subject will be treated in a forthcoming paper by the writer. The present paper was written without my being aware of the work of Ascoli.

that $A_{i,2}$ is the only limit-point of the set. Let $[a]_{i,2}$ be the set of arcs of $[a]$ on which lie the points $P'_{[a]_i,2}$. For $j = 3$ we now select an infinite subset $P'_{[a]_i,3}$ of $P_{[a]_i,3}$ which has only one limit-point $A_{i,3}$, and all of whose points lie on a subset $[a]_{i,3}$ of the set $[a]_{i,2}$.* We note that $A_{i,2}$ is a limit-point of that subset of $P_{[a]_i,2}$ which lies on arcs of $[a]_{i,3}$.

In general, for any fixed value of j we select an infinite subset $P'_{[a]_i,j}$ of $P_{[a]_i,j}$ which has only one limit-point $A_{i,j}$, and which lies on arcs of $[a]_{i,j-1}$. That subset of $[a]_{i,j-1}$ on which lie points of $P'_{[a]_i,j}$ we denote by $[a]_{i,j}$.

Finally we denote by $[a]_i$ the set of arcs of $[a]$ which are arcs of all the sets $[a]_{i,j}$ $(j = 2, \ldots, m_i - 1)$. The points of $P'_{[a]_i,j}$ $(j = 2, \ldots, m_i - 1)$ which lie on arcs of $[a]_i$ we denote by $P_{[a]_i,j}$. Clearly $A_{i,j}$ is a limit-point of $P_{[a]_i,j}$ for all admitted values of j.

Then on each arc, as $a_{i,1}$, of $[a]_i$ we have a set of m_i points $P_{a_{i,1},j}$ in the order

$$A = P_{a_{i,1},1}; \quad P_{a_{i,1},2}; \quad P_{a_{i,1},3}; \quad \ldots; \quad P_{a_{i,1},m_i-1}; \quad P_{a_{i,1},m_i} = B.$$

The m_i sets of points $P_{[a]_i,j}$ (i fixed; $j = 1, \ldots, m_i$) have the limit-points $A_{i,j}$ and no others.

In a similar manner for the same fixed i we now obtain a set of m_{i+1} sets of points $P_{[a]_{i+1},j}$ $(j = 1, \ldots, m_{i+1})$ and a set of arcs $[a]_{i+1}$ having the following properties:

(a) The set of arcs $[a]_{i+1}$ is a subset of $[a]_i$.

(b) $P_{[a]_{i+1},j}$ contains all those points of $P_{[a]_i,j}$ which lie on arcs of $[a]_{i+1}$.

(c) $P_{[a]_{i+1},j}$ consists of m_{i+1} points on each arc of $[a]_{i+1}$, and contains on each arc a point within each triangle of $[t]_{i+1}$ within which are points of the arc.

(d) For any fixed j, $P_{[a]_{i+1},j}$ has only one limit-point $A_{i+1,j}$.

(e) The set of points $A_{i+1,j}$ contains all points of the set $A_{i,j}$.

The notion *order* may now be associated as follows with the set of points $A_{i,j}$ $(i = 1, \ldots, \infty$; and for each value of i, $j = 1, \ldots, m_i)$. Let $A^{(1)}$ and $A^{(2)}$ be any two such points, each distinct from A and from B. Then there is a value of i, as $i = h$, such that for certain values of j, as $j = k$ and $j = l$, $A^{(1)}$ is

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* We note that a subset may contain all the elements of the original set; that is, the word subset does not necessarily mean proper subset. We also note that for fixed j the points $P_{[a]_i,j}$ may all coincide. In that case $A_{i,j}$ coincides with these points.

the limit-point of the $P_{[a]_h, k}$ and $A^{(2)}$ is the limit-point of $P_{[a]_h, l}$. Then the points $A, A^{(1)}, A^{(2)}, B$ are said to be in the same order as $A; P_{[a]_h, k}; P_{[a]_h, l}; B$ on the various arcs of $[a]_h$.

DEFINITION. *We now consider a set of points* $[A]$ *consisting of all points* $A_{i, j}$ $(i = 1, \ldots, \infty;$ *and for each value of* $i, j = 1, \ldots, m_i)$ *together with their limit points.*

1. THEOREM. *The set* $[A]$ *is a simple continuous arc connecting the points* A *and* B.

PROOF. (a) *The set* $[A]$ *is closed by definition.*

(b) *The set* $[A]$ *is connected.*

Suppose that $[A]$ contains two complementary subsets neither of which contains a limit-point of the other. Denote these sets together with their non-contained limit-points by $[A]_1$ and $[A]_2$ respectively. By the Heine-Borel Theorem we obtain two finite sets of triangles $[t]_1$ and $[t]_2$ such that every triangle of either set is entirely exterior to every triangle of the other set, and such that all points of $[A]_1$ lie within triangles of $[t]_1$ and all points of $[A]_2$ within triangles of $[t]_2$.

Since points of $A_{i, j}$ must lie within each triangle of $[t]_1$ and of $[t]_2$, and since for all values of i equal to or greater than a certain fixed number k only a finite number of arcs of $[a]_i$ can fail to contain points within a triangle which contains points of $A_{i, j}$, and since, further, an arc which contains points within triangles of $[t]_1$ and also within $[t]_2$ must contain points exterior to every triangle of both sets, it follows that only a finite number of the set $[a]_i$ $(i \gtrless k)$ can fail to contain points exterior to both sets of triangles. Denote by $[Q]$ the set of all points of $[a]_i$ $(i \gtrless k)$ which are exterior to the triangles of both sets, together with the limit-points of such points. Then $[A]_1$, $[A]_2$ and $[Q]$ are closed sets, no one containing a limit-point of the others. Hence we can place a finite set of triangles about the set $[Q]$ without enclosing any point of $[A]_1$ or $[A]_2$. Since an infinitude of arcs of every set $[a]_i$, $(i \gtrless k)$ contains points of Q, it follows by the definition of $P_{[a]_i, j}$ that for some value of i, as $i = h$, there will be a point of $P_{[a]_i, j}$ on every arc of $[a]_i$ which contains a point in $[Q]$, and hence there will be a limit-point of such points, that is, a point of $A_{i, j}$ which does not lie within a triangle of either of the sets $[t]_1$ and $[t]_2$, which is contrary to the assumption that all points of $[A]$ are points of one or the other of the sets $[A]_1$ and $[A]_2$.

(c) *The set* $[A]$ *contains no connected subset containing* A *and* B.

We show first that if a point $A^{(1)}$ of $A_{i,j}$ other than A or B is removed from $[A]$ the remaining set is not connected. This is done by showing that the set of points of $A_{i,j}$ between A and $A^{(1)}$ have no limit-point other than $A^{(1)}$ in common with the points of $A_{i,j}$ between $A^{(1)}$ and B. Suppose there is such common limit-point Q. About Q set a triangle t_1 of which $A^{(1)}$ is an exterior point, and about $A^{(1)}$ set a triangle t_2 of which no point of t_1 is an interior point. About Q set another triangle t_3 such that by the uniform continuity of the curves of $[a]$ with respect to the set no curve contains a point of t_1 between points of t_3. Since $A^{(1)}$ is a limit-point of a certain set of points $P_{[a]_i, j}$ (for fixed j), it follows that there must be some arc which contains interior points of t_2 between points of t_3, and which therefore contains points of t_1 between points of t_3, which contradicts the properties assumed for t_1 and t_3. Hence the subset remaining when a point of $A_{i,j}$ is removed from $[A]$ is not connected. Consider now any point $\bar{A}$ of $[A]$ but not of $A_{i,j}$. By the preceding, the point $\bar{A}$ distinguishes the points of $A_{i,j}$ into two classes $[A]'$ and $[A]''$ such that $\bar{A}$ is a limit-point of points of $A_{i,j}$ between any point of $[A]'$ and the point B, and also of such points between any point of $[A]''$ and A. Now if possible let Q be a common limit-point of $[A]'$ and $[A]''$ other than $\bar{A}$. As above, set triangles about $\bar{A}$ and Q, neither containing as interior point a point of the other, when the argument to show that Q is not a limit-point of both $[A]'$ and $[A]''$ is like that given above.

2. THEOREM. *For every entirely open set R containing all points of $[A]$ there is a value of i, $i = k$, such that for all values of $i \geqq k$ only a finite number of arcs of $[a]_i$ fail to lie entirely within R.*

PROOF. Set about the points of $[A]$ a finite number of triangles $[t]_1$ every interior point of which is a point of R. About a point A_1 of $[A]$, within a triangle t_1 of $[t]_1$, set a triangle $t^{(1)}$ within t_1 such that no arc of $[a]$ contains points on t_1 between points on $t^{(1)}$; and similarly for every point of $[A]$. Then we obtain a finite subset $[t]'$ of m of these triangles enclosing all points of $[A]$. These triangles may be ordered so that A lies within $t^{(1)}$ and B lies within $t^{(m)}$, and such that $t^{(i)}$ has interior points in common with both $t^{(i-1)}$ and $t^{(i+1)}$ $(i = 2, \ldots, m-1)$. Let $A^{(1)}$ be a point of $A_{i,j}$ within both $t^{(1)}$ and $t^{(2)}$, and in general $A^{(i)}$ a point of $A_{i,j}$ within both $t^{(i)}$ and $t^{(i+1)}$. Then there is a value of i, $\bar{i}$, for which all points $A^{(i)}$ $(i = 1, \ldots, m)$ are limit-points of $P_{[a]_i, j}$. Then only a finite number of arcs of the set $[a]_i$ contain points exterior to $[t]_1$.

DEFINITION. *The arc specified in the preceding theorem is said to be a limit-arc of the set, and the type of approach specified in (2) is called uniform approach.*

We now summarize the preceding in the following theorem :

3. THEOREM. *If* [a] *is a set of simple continuous arcs connecting two points A and B and lying in a closed region R, and if the arcs are uniformly continuous with respect to the whole set, then there is a continuous arc in R connecting A and B which is a limit-arc of the set* [a] *and which is approached uniformly by a certain subset of* [a].

§ 8. *Concerning the Existence of Minimizing Curves.*

(We now use the usual metric hypotheses of geometry and the Cartesian correspondence between points in a plane and the pairs of real numbers.) Consider a function $f(x, y)$ defined over a closed region R, and continuous and positive in that region. Let a_1 be an arc of finite length lying in R and connecting two of its points A and B. Let $A, P_1, \ldots, P_i, \ldots, P_n = B$ be a set of n points on a_1 lying in order on it from A to B, as indicated by the notation. Denote by $[\sigma_i]$ the length of the set of chords $A P_1, P_1 P_2, \ldots, P_{i-1} P_i$; and let ξ_i be the values of $f(x, y)$ at the points P_i. Denote by Δ the length of the longest segment of $[\sigma_i]$. Then

$$a_1 \int_A^B f(x, y) = L \sum_{\Delta \doteq 0} \sigma_i \cdot \xi_i \qquad (i = 1, \ldots, n).$$

This limit will necessarily exist and be finite if a_1 is continuous and of finite length and $f(x, y)$ is continuous.

In this manner a definite positive number $N(a)$ is associated with every arc a of finite length lying in R and connecting A and B. Consider now the set $[a]$ of all continuous arcs of finite length lying in R and connecting A and B. The lower bound $\underline{B}$ of the set of numbers $[N(a)]$ is greater than zero; *i. e.*, it is greater than or equal to the distance from A to B multiplied by the minimum of $f(x, y)$ in R. Select an infinite sequence of arcs $\{a_i\}$ such that the sequence of numbers $\{N(a_i)\}$ is non-oscillating decreasing with the limit $\underline{B}$.

1. THEOREM. *The arcs of the sequence* $\{a_i\}$ *satisfy the condition of uniform continuity with respect to the set of arcs* $\{a_i\}$ *(see* (3), §7),*and hence have at least one limit-curve.*

PROOF. Consider any point P of R. If there exists a neighborhood of P within which are points of only a finite number of arcs of $\{a_i\}$, then the condition is a direct consequence of the continuity of each arc. If there is an infinitude of arcs of $\{a_i\}$ containing points within every neighborhood of P, set any

triangle t_1 about P. Let the shortest distance from P to t_1 be d_1, and let M and m be the maximum and minimum respectively of $f(x, y)$ in R. Set about P a triangle t_2 with shortest distance d_2 from P to t_2 such that $d_2\, M < \dfrac{d_1\, m}{2}$.

Since $\underset{i\doteq\infty}{L}\ \{N(a_i)\} = \underline{B}$, it follows that for some value of i, as i_1, every arc a_{i_1+k} $(k = 0, \ldots, \infty)$ must fail to contain points of t_1 between points of t_2. Any triangle t_3 within t_2 and enclosing P which satisfies the condition for the finite set of arcs $a_1, \ldots, a_{i_1}$ must therefore satisfy the condition for the whole sequence $\{a_i\}$. Hence, by (3), § 7, there is a limit-arc $\bar{a}$ of the sequence $\{a_i\}$.

2. THEOREM. *The lengths of the arcs $\{a_i\}$ have a finite upper bound, and their limit-arc $\bar{a}$ is finite in length.*

PROOF. If the lengths of the arcs formed an unbounded set, the integrals would be an unbounded set, inasmuch as the integral of each curve is equal to or greater than its length multiplied by m. Let $\bar{B}$ be the upper bound of the lengths of the arcs of $\{a_i\}$. If $\bar{a}$ is of infinite length, we can find a set of points $P_1, P_i, \ldots, P_n$ on it such that $\overset{i=n}{\underset{i=1}{\Sigma}}\ \sigma_i$ shall be greater than $\bar{B} + d$, where d is any preassigned number. About each point P_i set a circle c_i of radius r, where $r < \dfrac{d}{2n}$ (n being the number of points P_i) or $2nr < d$. By (3), § 7, there is an arc $a^{(k)}$ of $\{a_i\}$ which contains points within every circle c_1. Hence the arc $a^{(k)}$ can not be less than $\Sigma \sigma_i$ by more than $2nr$. That is, it is greater than $\bar{B}$, which is contrary to the definition of $\bar{B}$.

DEFINITION. *Any number of a set such that there is no number of the set less than it, is called an "absolute minimum" of the set. A curve $\bar{a}$ of a set $[a]$, such that $N(\bar{a})$ is an absolute minimum of the set of numbers $[N(a)]$, is called a minimizing curve of the set.*

3. THEOREM. *If $f(x, y)$ is a continuous positive function defined over a closed, bounded set of points R, and if there exists an arc of finite length lying in R and connecting A and B, then there exists at least one such arc $\bar{a}$ in R for which $N(\bar{a})$ is an absolute minimum.*

PROOF. Suppose $\bar{a}$ is not a minimizing curve. Then there is some fixed positive number d such that, within every entirely open set containing $\bar{a}$, there is an arc $a^{(k)}$ such that $N(\bar{a}) - N(a^{(k)}) > d$. Let $A = P_1, \ldots, P_i, \ldots, P_n = B$ be a set of points P_i on $\bar{a}$, and σ_i the length of the corresponding chords such that

$$|\Sigma \sigma_i \cdot \xi_i - N(\bar{a})| < \frac{d}{4}.$$

By a well-known property of continuous curves, there exists a positive number r, such that, if circles c_i of radius r are described about P_i as centers, there are no points of the arcs $A\,P_{i-1}$ and $P_{i+1}\,A$ within the circle c_i, and, further, such that each circle is entirely exterior to every other. Within each circle c_i place a concentric circle c_i^* with radius $\dfrac{r}{2}$. From P_i trace the arc $\bar{a}$ towards A until first meeting c_i^* in a point P_i'. Similarly trace $\bar{a}$ from P_i towards B until first meeting c_i^* in P_i''. Consider the resulting set of arcs $A\,P_1'$, ..., $P_{i-1}''\,P_i'$, $P_i''\,P_{i+1}'$, ... (leaving out the arc $P_i'\,P_i''$). Between any two of these arcs there is a minimum distance. Let d_1 be the smallest of these distances. Using a circle whose radius r_1 is less than $\dfrac{d_1}{2}$, trace a neighborhood of $\bar{a}$ by letting the center of the circle pass over $\bar{a}$ from A to B. Denote this neighborhood of $\bar{a}$ by $R^{(1)}$. The circles c_i divide that part of R_1 which is exterior to them into a set of n regions R_i lying in order about $\bar{a}$ from A to B. In any one of these regions R_i there is a definite difference between the maximum and minimum values of $f(x, y)$. Denote by v_i the difference between the maximum and minimum values of $f(x, y)$ in R_{i-1}, R_i and R_{i+1} and the two circles c_i and c_{i+1}, and let V be the greatest of these values of v_i. Denote by l the length of $\bar{a}$ and by l_i the length of the arcs $P_{i-1}\,P_i$.

Let $a^{(k)}$ be any arc of $\{a_i\}$ within $R^{(1)}$. Then $a^{(k)}$ must contain points within each circle c_i. Construct a broken line $[\sigma_j^{(k)}]$ with vertices on $a^{(k)}$, having the following properties:

(a) $|\Sigma\,\sigma_j^{(k)}\cdot\xi_j^{(k)} - N(a^{(k)})| < \dfrac{d}{4}$.

(b) There is a vertex $P_i^{(k)}$ of $\sigma_j^{(k)}$ within each circle c_i.

Denote by $l_i^{(k)}$ the length of the broken line of $\sigma_j^{(k)}$ connecting $P_{i-1}^{(k)}$ and $P_i^{(k)}$. We note that $\sigma_i \lessgtr l_i$. Then σ_i can not exceed $l_i^{(k)}$ by more than $4\,r$, and hence $\sigma_i \cdot \xi_i$ can not exceed $\sigma_j^{(k)} \cdot \xi_j^{(k)}$ taken from $P_{i-1}^{(k)}$ to $P_i^{(k)}$ by more than $l_i\,V + 4r M$, where M is the maximum of $f(x, y)$ in R. Hence $\Sigma\,\sigma_i \cdot \xi_i$ can not exceed $\Sigma\,\sigma_j^{(k)} \cdot \xi_j^{(k)}$ by more than

$$V\Sigma\,l_i + 4\,n\,r\,M \quad \text{or} \quad V\,l + 4\,n\,r\,M.$$

In this expression l and M are fixed. V and n are fixed simultaneously and r is fixed independently of V and n. The processes are as follows: First impose on the points P_i of $\bar{a}$ the further condition that the maximum of the differences

between the maximum and minimum of $f(x, y)$, on an arc consisting of any three of the consecutive arcs into which P_i divides $\bar{a}$, shall be $V' < \dfrac{d}{8\,l}$. Then we can impose upon r_1 (the radius of the circle which traces out the region R_1) the additional condition that the variation V described above shall not be greater than $2\,V'$, whence $V < \dfrac{d}{4\,l}$ or $Vl < \dfrac{d}{4}$. We now impose upon r the further condition that $r < \dfrac{d}{16\,n\,M}$ (note that n is fixed when the points P_i are determined) or $4\,n\,r\,M < \dfrac{d}{4}$. Then

$$Vl + 4\,n\,r\,M < \frac{d}{2}.$$

Since, therefore, $\Sigma \alpha_i \cdot \xi_i$ can not exceed $\Sigma \sigma_j^{(k)} \cdot \xi_j^{(k)}$ by $\dfrac{d}{2}$, and since

$$|\Sigma \sigma_j^{(k)} \cdot \xi_j^{(k)} - N(a^{(k)})| < \frac{d}{4},$$

$$|\Sigma \sigma_i \cdot \xi_i - N(\bar{a})| < \frac{d}{4},$$

it follows that $N(\bar{a})$ can not exceed $N(a^{(k)})$ by d, which proves our theorem.

This theorem covers a special case of the general problem of existence of solutions in the calculus of variations. There the problem is to minimize the integral $\int f(x, y, y')\,dx$, whereas in the present theorem y' is not present. However, all cases where the expression to be minimized can be written in the form $\int f(x, y) \sqrt{1 + y'^2}\,dx$ come under the case treated here; *i. e.*, where the y' enter simply to involve the length of arc as a multiplicative factor. Thus the existence of a *shortest distance* between any two points in a closed connected region (if any arc of finite length in the region connects them) and the existence of a minimum surface of revolution follow from this theorem. It should be noted that no assumption, other than that of finite length, as to the character of the curves is made. The theorem says that among *all* curves there is *a curve* which has the required property. Further, there is no assumption as to the character of the boundary of the region in which the curves lie. For the purpose of this discussion region may be any connected set whatever.

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The Involutorial Birational Transformation of the Plane, of Order 17.

By Virgil Snyder.

§ 1. Introduction.

1. It has been shown by Bertini [*] that there are four distinct types of birational (Cremona) transformations in the plane that can not be transformed into simpler types, that are involutorial. From considerations of the Cremona net, the images of the lines of the plane, these are found to be :

(a) Harmonic homology.

(b) Jonquières's (perspective) transformation.

(c) Geiser's transformation.

(d) A transformation of order 17.

Of these four types the first three were known before, but type (d) was new. No equations are given, and but few of its properties are enumerated. In a foot-note a method of making the transformation geometrically is described [†] and attributed to Cremona. The contents of the note have been amplified by Sturm, [‡] and a summary of the above results are given by Doehlemann without further discussion. [§] This transformation is closely connected with that of the locus of the ninth double point of a net of sextics having eight fixed double points. [||]

It is the purpose of this paper to derive the equations of the transformation of order 17, and to discuss some of its properties.

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[*] "Ricerche sulle trasformazioni univoche involutorie nel piano," *Ann. di Mat.*, Ser. II, Vol. VIII (1877), pp. 244–286.

[†] *l. c.*, p. 273.

[‡] *Lehre von den geometrischen Verwandtschaften*, Vol. IV (1908), pp. 105–107 (No. 829).

[§] *Geometrische Transformationen*, Vol. II (1908), pp. 165–174.

[||] Halphen: "Sur les courbes planes du sixième degré à neuf points doubles," *Bulletin Soc. Math. de France*, Vol. X (1882), pp. 162–172.

§ 2. *Analytic Discussion.*

2. A rational curve of order 6 has 10 double points, but they can not be assigned arbitrarily. Indeed, if nine double points be chosen arbitrarily, the sextic through them will consist of the cubic determined by them counted twice, hence not more than eight double points can be chosen arbitrarily. Since these conditions are all linear when the positions of the double points are given, it follows that the sextics having eight given double points form a triply infinite linear system. The ninth double point must belong to some locus. A pencil of cubics pass through the eight given points, and they have one residual basis-point, which can not be a double point on the sextic, for a cubic of the pencil can pass through any tenth point. If this tenth point be chosen on the sextic, the cubic determined by it would have 19 points of intersection with the sextic, hence would be a factor of it.

3. Let f be a sextic having nine double points, and c a cubic passing through them. Every sextic of the pencil

$$c^2 + \lambda f = 0$$

will have these same nine points for double points. Hence: *If nine points are double points on a proper sextic curve, through them can be passed a pencil of sextics, each having the nine points for double points.* The pencil of sextics can not be transformed into a simpler one by any Cremona transformation.

4. Let the eight given points be denoted by $P_1, P_2, \ldots, P_8$, and ϕ, ψ denote two cubics passing through them, and f denote a sextic having them all for double points. The system

$$\alpha \phi^2 + \beta \phi \psi + \gamma \psi^2 + \delta c = 0 \tag{1}$$

represents a triply infinite linear system of sextics having P_i for double points. From the fact that any sextic of the set is completely fixed by three additional points, it follows that the above equation comprises every sextic having P_i for double points.

Let P' be any point in the plane. The curves of (1) which pass through P' form a net, the equation of which may be written

$$\alpha (\phi^2 f' - \phi'^2 f) + \beta (\psi^2 f' - \psi'^2 f) + \gamma (\phi \psi f' - \phi' \psi' f) = 0,$$

wherein ϕ' etc. denote the result of substituting the coordinates of P' in ϕ etc. By equating to zero the coefficients of α, β, γ, we obtain

$$\frac{\phi^2}{\phi'^2} = \frac{\psi^2}{\psi'^2} = \frac{\phi \psi}{\phi' \psi'} = \frac{f}{f'},$$

from which we see that all the curves of the net pass through all the intersections of

$$\phi\psi' - \phi'\psi = 0,$$
$$\phi\psi f' - \phi'\psi'f = 0.$$
$$\left.\right\} \quad (2)$$

The two curves of (2) intersect in 18 points, of which 16 are in P_i and one at P'. The residual P'' is fixed when P' is given. Hence:

In a triply infinite linear system of sextic curves having eight common fixed double points, the curves of the net determined by a further basis-point P' will all pass through another fixed point P''.

Since the coordinates of P' and of P'' enter the equations in exactly the same way, the birational transformation defined by them must be involutorial. The equations can be found from (2) by solving for the residual point of intersection. The image of a straight line l is obtained by eliminating the coordinates x_i from its equation and from (2). From the resultant of order 24, $\phi'\psi'$ divides out as a factor, and the linear expression involving P'. Hence:

By our involutorial transformation straight lines go into rational curves of order 17.

The equations of the transformation are believed to be new; the order 17 was determined by Bertini from the relations connecting fundamental curves of a Cremona net and by Cremona by means of the depiction of a cubic surface on a plane. From the method of elimination it follows immediately that P_i is a sixfold point on c_{17}, the image of an arbitrary line. The fundamental curves are therefore sextics. Since each must be rational and since any two can intersect only in fundamental points, each has a triple point at the fundamental point to which it corresponds, and double points at the remaining basis-points. These curves uniquely determine the transformation.

§ 3. *Geometric Interpretation.*

5. We may obtain the same result by considering the sextic having a triple point at P_1 and double points at each of the other basis-points, for which the symbol

$$f_6\,(1^3\,2^2\,3^2\,4^2\,5^2\,6^2\,7^2\,8^2)$$

will be used. If for P' we take any point on this curve, the curve of the pencil $\phi\psi f'-\phi'\psi'f = 0$ remains fixed while the associated cubic has one parameter. Thus any cubic intersects f_6 in 17 fixed points, and the image of any point P' on f_6 is therefore P_1 or 1. Since all points P_i enter symmetrically, the Jacobian of the system is of order 48; hence the order of the transformation is 17.

6. The ninth basis-point of the pencil of cubics is not a fundamental point of the transformation, nor are the two residual intersections of ϕ with f or of ψ with f, since ϕ', ψ' divide out of the equations of the transformation. This fact illustrates a general principle which can be applied in a number of similar cases. Consider the pencil of lines $l + \lambda m = 0$ and of conics $c + \mu k = 0$. Any point P' will uniquely determine a line $lm' - l'm = 0$ and a conic $ck' - c'k = 0$. The line and conic intersect in P' and in one other point P'' uniquely fixed by P'. The pairs of points P', P'' determine an involution and define a birational transformation. Evidently the lines passing through (l, m) and any basis-point of the pencil of conics are simple fundamental lines belonging to the fundamental point through which they pass, respectively. In the same way we may consider the fundamental curve belonging to the point (l, m). The result is the conic of the pencil determined by (l, m). The configuration of fundamental curves being of order 6, the transformation is the Jonquières transformation of order 3, and may be defined by a cubic curve and a point P upon it. Any line through P cuts the curve in two points A_1, A_2. The image of any point P' is the harmonic conjugate of P' as to A_1, A_2 on the line PP'. The fundamental curve of P is its polar conic as to the cubic, and the fundamental lines are the tangents to the cubic from P.

7. Now suppose one conic of the pencil to be composite, one factor passing through (l, m). This requires that two basis-points lie on a straight line containing (l, m). This line now divides out as a factor. The fundamental conic is replaced by a straight line and the transformation becomes quadratic inversion.

8. If the pencil of conics is of the form

$$c + \mu l m = 0,$$

both l, m divide out as factors and no fundamental curves remain. The transformation is now harmonic homology with (l, m) as center and its polar as to c as axis.

The second case can be derived from the third by transforming through an inversion having one vertex at the center of the homology, and neither of the others on the axis.

9. Another illustration is furnished by the pencil of conics and a pencil of quartics, the latter having double points at three of the basis-points of the conics, but not passing through the fourth. The transformation is now of order 8, but reduces to 4 when the pencil of quartics is of the form $c_2 k_2 + \mu c_4 = 0$, when the pencil of conics is $c_2 + \lambda k_2 = 0$. The residual basis-point of the pencil of conics is not a fundamental point of the transformation.

10. Every involutorial birational transformation in the plane can be defined by two pencils of curves having a sufficient number of common basis-points to account for all but two intersections of a curve of one pencil and one of the other. By associating those curves of the two pencils which pass through a point P', the residual intersection P'' can be found by rational operations. Since every curve has a series g_2', it follows that such systems must always be hyperelliptic, elliptic or rational. From known theorems regarding the canonical form of hyperelliptic curves we again arrive at Bertini's theorem that all involutorial birational transformations can be transformed into one of the four types mentioned above. Interesting particular cases can be obtained by choosing the fundamental points of the transforming operator in special positions.

§ 4. *Curve of Invariant Points.*

11. Every Cremona transformation possesses a number of invariant points; when the transformation is involutorial these form a locus. This locus of invariant points must be distinguished from an invariant curve which has its points exchanged in pairs by the transformation. The former points will appear whenever $P' \equiv P''$. In our case this will occur when a cubic touches the associated sextic. Since a net of sextics pass through P_i, P', P'', it follows that at least one curve of the system has a double point at P', and is therefore a factor of the Jacobian of the net. After a few reductions this is found to be

$$f'(\phi \psi' - \phi' \psi)^2 \frac{\partial (\phi, \psi, f)}{\partial (x_1, x_2, x_3)} = 0.$$

Since f is a fixed curve and P' can be chosen at will, the factors f, $\phi \psi' - \phi' \psi$ may be rejected; hence the proper locus is

$$J \equiv \frac{\partial (\phi, \psi, f)}{\partial (x_1, x_2, x_3)} = 0.$$

Any point of J can be a ninth double point on a proper sextic of the system except the critical centers of the pencil $\phi + \lambda \psi = 0$. These points are double points on curves of the pencil of cubics; for them $P' \equiv P''$ and $J = 0$; but if k is a critical center the sextic having this point as a double point will meet the associated cubic in 20 points, that is, the sextic must degenerate.

The coordinates of these points satisfy the relations

$$\frac{\phi_1}{\psi_1} = \frac{\phi_2}{\psi_2} = \frac{\phi_3}{\psi_3},$$

wherein ϕ_i denotes differentiation of ϕ as to x_i.

12. The Jacobian J is of order 9; it evidently passes through P_j since f has a double point at each basis-point. It will now be shown that J has a triple point at each basis-point of the system. If we differentiate J as to x_i, we obtain:

$$J_i \equiv \begin{vmatrix} \phi_{1i} & \psi_1 & f_1 \\ \phi_{2i} & \psi_2 & f_2 \\ \phi_{3i} & \psi_3 & f_3 \end{vmatrix} + \begin{vmatrix} \phi_1 & \psi_{1i} & f_1 \\ \phi_2 & \psi_{2i} & f_2 \\ \phi_3 & \psi_{3i} & f_3 \end{vmatrix} + \begin{vmatrix} \phi_1 & \psi_1 & f_{1i} \\ \phi_2 & \psi_2 & f_{2i} \\ \phi_3 & \psi_3 & f_{3i} \end{vmatrix}.$$

The first two determinants vanish at P_j since $f_i = 0$. In the third determinant, multiply the first row by x_1, the second by x_2, and the third by x_3, and add the products for a new third row. The result is:

$$x_3 J_i = \begin{vmatrix} \phi_1 & \psi_1 & f_{1i} \\ \phi_2 & \psi_2 & f_{2i} \\ 3\phi & 3\psi & 5f_i \end{vmatrix} = 0,$$

which shows that P_j is a double point on J.

Now compute the second derivatives as to x_i, x_k and retain only those determinants which do not vanish for the reasons mentioned above. We may write:

$$J_{ik} = (\phi_{1i}\,\psi_1\,f_{1k}) + (\phi_1\,\psi_{1i}\,f_{1k}) + (\phi_1\,\psi_1\,f_{1ik}) + (\phi_{1k}\,\psi_1\,f_{1i}) + (\phi_1\,\psi_{1k}\,f_{1i}).$$

By performing the same operations as before, we have:

$$x_3 J_{ik} = 2\phi_i \begin{vmatrix} \psi_1 f_{1k} \\ \psi_2 f_{2k} \end{vmatrix} - 2\psi_i \begin{vmatrix} \phi_1 f_{1k} \\ \phi_2 f_{2k} \end{vmatrix} + 4f_{ik} \begin{vmatrix} \phi_1 & \psi_1 \\ \phi_2 & \psi_2 \end{vmatrix} + 2\phi_k \begin{vmatrix} \psi_1 f_{1i} \\ \psi_2 f_{2i} \end{vmatrix} - 2\psi_k \begin{vmatrix} \phi_1 f_{1i} \\ \phi_2 f_{2i} \end{vmatrix},$$

which may be written in the form:

$$2 \begin{vmatrix} \phi_1 & \psi_1 & f_{1k} \\ \phi_2 & \psi_2 & f_{2k} \\ \phi_i & \psi_i & f_{ik} \end{vmatrix} + 2 \begin{vmatrix} \phi_1 & \psi_1 & f_{1i} \\ \phi_2 & \psi_2 & f_{2i} \\ \phi_i & \psi_i & f_{ii} \end{vmatrix}.$$

When $i = 1$ or $i = 2$, we have two identical rows; when $i = 3$, we can make the elements of the third row zero by performing the same operations as before. We now have the theorem:[*]

The locus of coincident points in the involutorial transformation of order 17 is a curve of order 9, having a triple point at each basis-point. It is the locus of the ninth double point of a sextic curve having double points at eight given points.

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[*] E. C. Valentiner: "Nogle Sätninger on visse algebraiske Kurven," *Zeuthen's Zeitschrift*, Ser. IV, Vol. V (1881), pp. 88–91.

13. A curve of order 9, having triple points at eight given points belongs to a linear system having six degrees of freedom. It may therefore be defined by an equation of the form

$$(\phi, \psi)_3 + (\phi, \psi)_1 f + k J = 0,$$

wherein $(\phi, \psi)_i$ is a binary form of order i, and k is a constant. In general, the curves of this system which pass through a point P' will define a system of five degrees of freedom and having no further point in common. If, however, the P' be chosen so that the curves defined by the equations

$$\phi \psi' - \phi' \psi = 0, \qquad \phi^2 f' - \phi'^2 f = 0, \qquad \phi^3 J' - \phi'^3 J = 0$$

meet in a point, then all the curves of the system of order 9 will also pass through another point P''. The locus of P' is a curve of order 18, having sixfold points at the basis-points P_j. The point P'' lies on the same curve, and the system of curves defines an involutorial transformation which leaves the curve as a whole invariant. Among the curves of order 9 having threefold points at P_j are those having a ninth triple point. This can not be chosen arbitrarily, for the conditions are all linear and one such curve is the cubic determined by the nine points, taken three times. After a ninth point has been found which is a triple point of a proper curve of the system, then every curve of the pencil

$$\phi^3 + \lambda c_9 = 0$$

will have nine triple points. This pencil can not be reduced to a simpler one by birational transformations. In this way a general elliptic system of curves of order $3r$ having nine r-fold points can be found. Of these, eight points may be chosen arbitrarily, and the ninth lies on a definite locus. If this point lies on J, every curve of the system remains invariant under our transformation T_{17}.

14. An arbitrary straight line c_1 goes into c_{17} by T_{17}. Since c_1 cuts J_9 in nine points through which the image c_{17} must pass, c_1 cuts c_{17} in eight other points, which must be arranged in pairs, since the transformation is involutorial. Caporali [*] has suggested the word class to define the number of pairs of points on an arbitrary straight line. *The involutorial transformation of order 17 is of class 4.* Apart from the basis-points, c_{17} and J_9 have only these nine points in common. Conversely, given any c_{17} having sixfold points at all the basis-points P_j. It will intersect J_9 in nine points which always lie on a straight line, the image of c_{17} in T_{17}.

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[*] "Sulle trasformazioni univoche piane involutorie," *Memorie di Geometria*, Naples, 1888, pp. 116–125.

§ 5. *Transformation through Quadratic Inversions.*

15. If T_{17} be transformed through a quadratic inversion having all its vertices at basis-points, it will go into one of similar form, but if one of the vertices be an arbitrary point P_9, a new form will result. Let I_{129} denote an inversion having vertices at P_1, P_2, P_9. It will be convenient to indicate the basis-points by their subscripts. Let $\bar{9}$ be the image of 9 under T_{17} and ξ be the image of $\bar{9}$ under I_{129}. By using the notation $c_1 I_{129} c_2 (129)$ to indicate that the image of a straight line in I_{129} is a conic through the vertices 1, 2, 9, we may write

$$c_2 (129)\, T_{17}\, c_{22} (1^7\, 2^7\, 3^8\, 4^8\, 5^8\, 6^8\, 7^8\, 8^8\, \bar{9}^1)\, I_{129}\, \bar{c}_{30} (1^{15}\, 2^{15}\, 3^8\, 4^8\, 5^8\, 6^8\, 7^8\, 8^8\, 9^8\, \xi^1)$$

and

$$I_{129}\, T_{17}\, I_{129} = T_{30}.$$

The configuration of fundamental curves becomes:

$$1: \quad c_{15} (1^8\, 2^7\, 3^4\, 4^4\, 5^4\, 6^4\, 7^4\, 8^4\, 9^4\, \xi^1),$$
$$2: \quad c_{15} (1^7\, 2^8\, 3^4\, 4^4\, 5^4\, 6^4\, 7^4\, 8^4\, 9^4\, \xi^1),$$
$$3: \quad c_8 (1^4\, 2^4\, 3^3\, 4^2\, 5^2\, 6^2\, 7^2\, 8^2\, 9^2),$$
$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots,$$
$$9: \quad c_8 (1^4\, 2^4\, 3^2\, 4^2\, 5^2\, 6^2\, 7^2\, 8^2\, 9^3),$$
$$\xi: \quad c_1 (1\, 2).$$

The invariant curve J_9 becomes $J_{12}(1^6\, 2^6\, 3^3 \ldots 9^3)$. If, however, the point 9 be chosen on J_9, the new transformation is of order 29, obtained from the preceding case by removing the factor $c_1 (12)$ from c_{30}, each c_{15} and c_1, as well as suppressing ξ in the preceding form. J is now of order 11.

The class of T_{30} is 9, and so is T_{29}. *The class of an involution is not invariant under birational transformation.*

16. Geometrically, T_{30} can be obtained as follows: Let ϕ_4, ψ_4 be two quartic curves having double points at 1, 2, and passing simply through 3, ..., 9. Let f_8 be an octic having fourfold points at 1, 2 and double points at 3, ..., 9. Every quartic of the pencil $\phi_4 + \lambda \psi_4 = 0$ will have the same property as those described, as will also every octic of the pencil $\phi_4 \psi_4 + \mu f_8 = 0$. A quartic and an octic will intersect in two variable points, either of which uniquely determines the other. In addition to the basis-points, the images of the lines of the plane have one other point in common, in order to have two degrees of freedom and have only one variable point of intersection. The whole configuration can

be obtained from the preceding one by transforming the two systems of curves through the quadratic inversion. The necessary and sufficient condition that T_{29} results is that all nine basis-points can be double points on a proper sextic curve.

§ 6. *Particular Cases.*

17. If three of the basis-points, say 1, 2, 3, lie on a straight line, it must be a factor of c_{17} and of the fundamental curves of 1, 2, 3. The configuration becomes:

$$c_1\, T_{16} \cdot c_{16}\, (1^5\, 2^5\, 3^5\, 4^6\, 5^6\, 6^6\, 7^6\, 8^6),$$
$$1:\quad c_5\, (1^2\, 2^1\, 3^1\, 4^2\, 5^2\, 6^2\, 7^2\, 8^2),$$
$$2:\quad c_5\, (1^1\, 2^2\, 3^1\, 4^2\, 5^2\, 6^2\, 7^2\, 8^2),$$
$$3:\quad c_5\, (1^1\, 2^1\, 3^2\, 4^2\, 5^2\, 6^2\, 7^2\, 8^2),$$

and the others as in the general case. The proper curve of invariant points is now:

$$J_8\, (1^2\, 2^2\, 3^2\, 4^3\, 5^3\, 6^3\, 7^3\, 8^3).$$

These same results can be derived directly from the equations. Let c_1 denote the line joining 1, 2, 3 and c_2 the conic containing the other five basis-points. For ϕ_3 we may now write $c_1 c_2$ and for f_6 we can take $c_1 c_5$, wherein c_5 is defined by the symbol $c_5\, (1^1\, 2^1\, 3^1\, 4^2\, 5^2\, 6^2\, 7^2\, 8^2)$.

The pencils analogous to (2) are now:

$$\psi_3'\, c_1\, c_2 - c_1'\, c_2' \cdot \psi_3 = 0,$$
$$c_2'\, \psi_3' \cdot c_5 - c_5' \cdot c_2\, \psi_3 = 0.$$

To determine the fundamental curves, consider $c_5\, (1^2\, 2^1\, 3^1\, 4^2\, 5^2\, 6^2\, 7^2\, 8^2)$. If P' be chosen anywhere on this curve, P'' is evidently at 1, hence c_5 is the fundamental curve corresponding to 1. Similarly for the points 2 and 3. To determine the fundamental curve of 4, associate curves of the two pencils which touch each other at 4. The locus of the residual intersection is the required fundamental curve. It will have a fourfold point at 4, a double point at 1, 2 and 3 and triple points at 5, 6, 7, 8; it is of order 8. The locus must also pass through the residual intersection of $c_2\, (4, 5, 6, 7, 8)$ and ψ_3. As the former has 17 points on c_8, it must be a factor; the other component is $c_5\, (1^2\, 2^2\, 3^2\, 4^3\, 5^2\, 6^2\, 7^2\, 8^2)$, as in the general case. By performing the operations indicated in the definition of J, $c_1\,(1\,2\,3)$ appears as a factor, leaving J_8 as defined above.

18. If 1, 2, 3 lie on c_1 and 4, 5, 6 on k_1, then both lines appear twice in c_{17}. The transformation is now of order 13 and has the symbol

$$c_1\, T_{13}\, c_{13}\, (1^4\, 2^4\, 3^4\, 4^4\, 5^4\, 6^4\, 7^2\, 8^6).$$

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The fundamental curves become

$$1: \quad c_4 \left(1^2\, 2^1\, 3^1\, 4^1\, 5^1\, 6^1\, 7^2\, 8^2\right),$$

and similarly for $2, \ldots, 6$; those for $7, 8$ are as in the general case. Since $\phi_3 = c_1\, k_1\, l_1$, $f_6 = c_1\, k_1\, c_4$, the equations of the generating pencils may be written in the form

$$\psi_3' \cdot c_1\, k_1\, l_1 - c_1'\, k_1'\, l_1' \cdot \psi_3 = 0,$$
$$l_1'\, \psi_3' \cdot c_4 - c_4' \cdot l_1\, \psi_3 = 0.$$

The locus of coincident points has the symbol

$$J_7 \left(1^2\, 2^2\, 3^2\, 4^2\, 5^2\, 6^2\, 7^3\, 8^3\right).$$

The transformation is now of class 3.

19. In case two of the basis-points approach coincidence, forming a tacnode, the transformation can be reduced to that having three collinear basis-points by transforming through a quadratic inversion whose vertices are the tacnode and any other two basis-points. Conversely, from the three collinear points we can obtain the tacnode. Thus, if $1, 2, 3$ are collinear, we may write

$$c_1\, I_{124}\, c_2\, (1\ 2\ 4)\ T_{16}\, c_{16}\, \left(1^5\, 2^5\, 3^6\, 4^5\, 5^6\, 6^6\, 7^6\, 8^6\right)\, I_{124}\, c_{17}\, \left(1^6\, 2^6\, \overline{3\ 4}{}^6\, 5^6\, 6^6\, 7^6\, 8^6\right),$$

in which $3, 4$ form a tacnode.

If in the same case we transform through I_{456}, we have

$$c_1\, I_{456}\, c_2\, (4\ 5\ 6)\ T_{16}\, c_{14}\, \left(1^4\, 2^4\, 3^4\, 4^5\, 5^5\, 6^5\, 7^6\, 8^6\right)\, I_{456}\, c_{13}\, \left(1^4\, 2^4\, 3^4\, 4^4\, 5^4\, 6^4\, 7^6\, 8^6\right),$$

the general Geiser transformation results.
in which the points $1, \ldots, 6$ lie on a conic. A particular case was met with before when $1, 2, 3$ lie on c_1 and $4, 5, 6$ on k_1. The configuration of fundamental curves is the same as given for the two lines.

If finally T_{13} be transformed through I_{678}, thus:

$$c_1\, I_{678}\, c_2\, (6\ 7\ 8)\ T_{13} \cdot c_{10}\, \left(1^3\, 2^3\, 3^3\, 4^3\, 5^3\, 6^2\, 7^7\, 8^8\right)\, I_{678}\, c_8\, \left(1^3\, 2^3\, 3^3\, 4^3\, 5^3\, 7^3\, 8^3\right),$$

the general Geiser transformation results.

20. Thus the three cases of a tacnode, three collinear points, and six points on a conic can all be transformed into each other by proper quadratic inversions, and each is equivalent to the general Geiser transformation (c). Further particularizations of the basis-points in T_{17} will therefore reduce to particular cases of the Geiser T_8. As the latter have already been treated at length[*] they need not be considered here.

CORNELL UNIVERSITY, *August*, 1910.

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[*] Snyder: "Conjugate Line Congruences Contained in a Bundle of Quadric Surfaces," *Transactions Amer. Math. Soc.*, Vol. XI (1910), pp. 371–387.

On the Problem of Two Fixed Centres and Certain of Its Generalizations.

By Adam Miller Hiltebeitel.

The classic problem of two fixed centres may be stated thus: To determine the motion of a body attracted by two fixed Newtonian centres of force. It was first solved by Euler and since Euler generalizations of the problem have been made. In this paper are given the resolution of the most general two-centre problem of which the variables are separable and a discussion of the trajectories of a less general form of the problem.

PROBLEM: To study the motion of a material body M acted upon by five centres of force, K_1, K_2, K_3, K_4, K_5, which are the two real foci, the centre, and the two imaginary foci, respectively, of a system of confocal conics, by forces

$$R_1 = - m r_1 - \frac{m_1}{r_1^2}, \quad R_2 = - m r_2 - \frac{m_2}{r_2^2}, \quad R_3 = - m_3 r_3,$$

$$R_4 = - m' r_4 - \frac{m_4 + i m_5}{r_4^2}, \quad R_5 = - m' r_5 - \frac{m_4 - i m_5}{r_5^2},$$

respectively, and by an additional force always parallel to the line $K_4 K_5$ and varying inversely as the cube of the distance of the moving body M from the fixed plane perpendicular to $K_4 K_5$ at K_3, the centres K_3, K_4, K_5 being fixed, while the real foci K_1, K_2 are always in the meridian plane $K_4 M K_5$.

Let x, y_1, z_1 be the coordinates of the moving body M referred to a fixed system of rectangular axes whose origin is at K_3 and whose x-axis coincides with the line passing through the imaginary foci K_4, K_5, as in Fig. 1. Imagine a plane through the x-axis and the moving body M, and let the intersection of this moving plane with the fixed $y_1 z_1$-plane be YY'. This line will always lie in the meridian plane $K_4 M K_5$ and, as this plane rotates about the x-axis, YY' will rotate about the origin. Every point in this line will describe a circle in the $y_1 z_1$-plane, whose centre is K_3, and upon it the two real foci are taken. Of the five centres of force, therefore, K_3, K_4, K_5 are fixed, while the other two move on a circle whose centre is K_3.

The motion of the meridian plane, and therefore that of the centres K_1, K_2, is due to that element of the initial velocity of M which is perpendicular to the initial position of the meridian plane. But this element of the initial velocity not only causes the rotation of the meridian plane, but also influences the nature of the path described by the moving body in this plane. By the aid of the principle of areas the motion of M in the rotating plane can be completely separated from the motion of rotation about the x-axis, as Jacobi[*] has done for the simpler problem.

The perpendicular distances of M from the x-axis and the line YY' are designated by y and x respectively. Regarding XX' and YY' as a system of rectangular axes in the meridian plane, then x, y are the rectangular coordinates of M defining its position in this plane. If at the beginning of motion the plane $K_4 M K_5$ coincides with the fixed plane $X K_3 Y_1$, and during the time t describes the angle ϕ, as indicated in Fig. 1, then

$$y_1 = y \cos \phi, \qquad z_1 = y \sin \phi, \qquad y_1^2 + z_1^2 = y^2. \qquad (1)$$
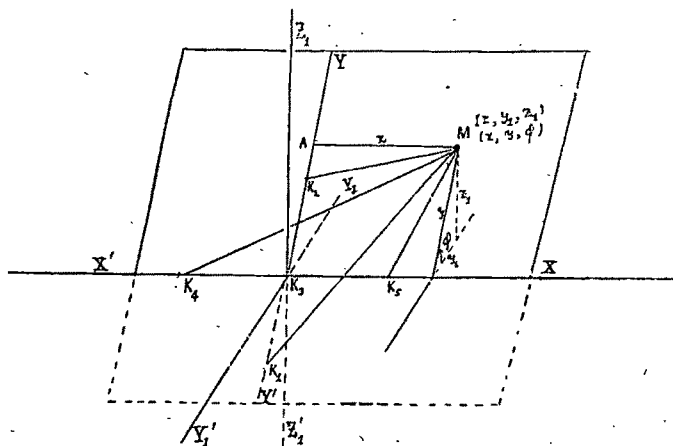
FIG. 1.

Let $2c$ denote the distance $K_1 K_2$ and $2ic$ the distance $K_4 K_5$. Let the rectangular coordinates of K_1, K_2 in the $Y_1 Z_1$-plane be $(-a, -b)$, (a, b), respectively, and the distances $K_1 M$, $K_2 M$, $K_3 M$, $K_4 M$, $K_5 M$ be respectively r_1, r_2, r_3, r_4, r_5. We now have:

$$a^2 + b^2 = c^2, \quad b y_1 - a z_1 = 0, \qquad (2)$$

$$r_1^2 = x^2 + (y_1 + a)^2 + (z_1 + b)^2, \quad r_2^2 = x^2 + (y_1 - a)^2 + (z_1 - b)^2, \qquad (3)$$

$$r_3^2 = x^2 + y_1^2 + z_1^2, \quad r_4^2 = (x + ic)^2 + y_1^2 + z_1^2, \quad r_5^2 = (x - ic)^2 + y_1^2 + z_1^2. \qquad (4)$$

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[*] *Vorlesungen über Dynamik*, No. 29. *Werke*, Supplement, pp. 221–231.

The form of the forces R_1, R_2, R_4, R_5 is accounted for by supposing that there are two masses at each of the corresponding centres, one acting inversely as the square of the distance and the other directly as the distance. At K_1, K_2 the masses acting inversely as the squares of the distances, denoted by m_1, m_2 respectively, may be any real quantities whatever, while the masses acting directly as the distances must be real and equal to each other. These latter masses are denoted by m. But at K_4, K_5 the masses acting inversely as the squares of the distances will be denoted by the conjugate complex quantities, $m_4 + i m_5$, $m_4 - i m_5$, i being the imaginary unit, and the masses acting as the direct distances may be any real quantity, as m'. Under these hypotheses as to the masses, the potential function will be real and the separation of the variables in the Hamilton-Jacobi partial differential equation can be effected by the aid of elliptic coordinates.

This problem includes, as particular cases, all the various forms of the two-centre problem. The form of the potential function given by Darboux[*] may be obtained by setting $m = m' = 0$, or by compounding the two forces $- m r_1$, $- m r_2$ into the single force $- 2 m r_3$ and suppressing the forces $- m' r_4$, $- m' r_5$.

These four forces together with the force $- m_3 r_3$ give rise to the following terms in the potential function:

$$- \tfrac{1}{2} m\, r_1^2 - \tfrac{1}{2} m\, r_2^2 - \tfrac{1}{2} m_3\, r_3^2 - \tfrac{1}{2} m'\, r_4^2 - \tfrac{1}{2} m'\, r_5^2 ;$$

and it can easily be shown that this expression is equivalent to

$$- (m + \tfrac{1}{2} m_3 + m')\, r_3^2 - c^2 (m - m').$$

Hence, even on retaining these forces, the potential function of this problem differs from that of Darboux's problem only by a constant term. Velde's[†] problem is obtained from the present problem by omitting the forces R_4, R_5 as well as that parallel to the x-axis, and by regarding the motion as in a fixed plane through the centres K_1, K_2, which are also supposed fixed. By simply omitting the forces R_4, R_5, we have the most general form of the two-centre

[*] *Archives Néerlandaises des Sciences*, Ser. 2, Vol. VI, pp. 371–376.

[†] *Programm der ersten höheren Bürgerschule zu Berlin*, 1889.

problem as formulated by Liouville.[*] Retaining only the forces R_1, R_2 and regarding the corresponding centres as fixed, we have the problem as extended by Lagrange;[†] and if, in addition, we suppress the terms $-mr_1$, $-mr_2$, in R_1, R_2 respectively, we have the original two-centre problem of Euler.[‡]

Differentiating equations (2), (3), (4), we have

$$a\,da + b\,db = 0, \qquad b\,dy_1 + y_1\,db - a\,dz_1 - z_1\,da = 0, \tag{5}$$

$$\left.\begin{array}{l} r_1\,dr_1 = x\,dx + (y_1 + a)\,dy_1 + (y_1 + a)\,da + (z_1 + b)\,dz_1 + (z_1 + b)\,db, \\ r_2\,dr_2 = x\,dx + (y_1 - a)\,dy_1 - (y_1 - a)\,da + (z_1 - b)\,dz_1 - (z_1 - b)\,db, \end{array}\right\} \tag{6}$$

$$\left.\begin{array}{l} r_3\,dr_3 = x\,dx + y_1\,dy_1 + z_1\,dz_1, \\ r_4\,dr_4 = (x + ic)\,dx + y_1\,dy_1 + z_1\,dz_1, \\ r_5\,dr_5 = (x - ic)\,dx + y_1\,dy_1 + z_1\,dz_1. \end{array}\right\} \tag{7}$$

But

$$(y_1 + a)\,da + (z_1 + b)\,db = \frac{z_1}{b}(a\,da + b\,db) + a\,da + b\,db = 0,$$

$$-(y_1 - a)\,da - (z_1 - b)\,db = -\frac{z_1}{b}(a\,da + b\,db) + a\,da + b\,db = 0;$$

and therefore the equations (6) become

$$\left.\begin{array}{l} r_1\,dr_1 = x\,dx + (y_1 + a)\,dy_1 + (z_1 + b)\,dz_1, \\ r_2\,dr_2 = x\,dx + (y_1 - a)\,dy_1 + (z_1 - b)\,dz_1. \end{array}\right\} \tag{8}$$

It will now be necessary to express the distances in terms of the variables x, y. This can be done without difficulty by aid of the relations (1) and (2), and we have

$$\left.\begin{array}{l} r_1^2 = x^2 + (y + c)^2, \quad r_2^2 = x^2 + (y - c)^2, \quad r_3^2 = x^2 + y^2, \\ r_4^2 = (x + ic)^2 + y^2, \quad r_5^2 = (x - ic)^2 + y^2. \end{array}\right\} \tag{9}$$

[*] *Journ. de Math.*, Vol. XI, pp. 345–378; Vol. XII, pp. 410–444; Vol. XIII, pp. 34–37. *Connaissance de Temps pour* 1849, pp. 255–256.

[†] *Miscellanea Taurinensia*, Vol. IV, pp. 118–2∃3: *Œuvres*, Vol. II, pp. 67–121; *Mécanique Analytique*, 1st Ed., pp. 262–286; 2nd Ed., Vol. II, pp. 108–121; *Œuvres*, Vol. XII, pp. 101–114.

[‡] *Nov. Comm. Acad. Imp. Petropolitanae*, Vol. X, pp. 207–242; Vol. XI, pp. 152–184. *Mémoires de l'Acad. de Berlin*, Vol. XI, pp. 228–249.

Taking the mass M as the unit mass, the differential equations of motion may now be written as follows:

$$
\begin{aligned}
\frac{d^2 x}{dt^2} &= - mx - mx - m_3 x - m'(x + ic) - m'(x - ic) - \frac{m_1 x}{r_1^3} - \frac{m_2 x}{r_2^3} \\
&\quad - \frac{(m_4 + im_5)(x + ic)}{r_4^3} - \frac{(m_4 - im_5)(x - ic)}{r_5^3} - \frac{\beta}{x^3}, \\
\frac{d^2 y_1}{dt^2} &= - m(y_1 + a) - m(y_1 - a) - m_3 y_1 - m' y_1 - m' y_1 \\
&\quad - \frac{m_1(y_1 + a)}{r_1^3} - \frac{m_2(y_1 - a)}{r_2^3} - \frac{(m_4 + im_5) y_1}{r_4^3} - \frac{(m_4 - im_5) y_1}{r_5^3}, \\
\frac{d^2 z_1}{dt^2} &= - m(z_1 + b) - m(z_1 - b) - m_3 z_1 - m' z_1 - m' z_1 \\
&\quad - \frac{m_1(z_1 + b)}{r_1^3} - \frac{m_2(z_1 - b)}{r_2^3} - \frac{(m_4 + im_5) z_1}{r_4^3} - \frac{(m_4 - im_5) z_1}{r_5^3}.
\end{aligned}
\tag{10}
$$

Multiplying equations (10) respectively by dx, dy_1, dz_1, adding and integrating, we obtain the vis viva integral

$$
\begin{aligned}
\tfrac{1}{2}\left\{ \left(\frac{dx}{dt}\right)^2 + \left(\frac{dy_1}{dt}\right)^2 + \left(\frac{dz_1}{dt}\right)^2 \right\} &= - \tfrac{1}{2} m r_1^2 - \tfrac{1}{2} m r_2^2 - \tfrac{1}{2} m_3 r_3^2 - \tfrac{1}{2} m' r_4^2 \\
&\quad - \tfrac{1}{2} m' r_5^2 + \frac{m_1}{r_1} + \frac{m_2}{r_2} + \frac{m_4 + im_5}{r_4} + \frac{m_4 - im_5}{r_5} + \tfrac{1}{2} \cdot \frac{\beta}{x^2} + h.
\end{aligned}
\tag{11}
$$

Now multiplying the second equation of (10) by z_1, and the third by y_1, and then subtracting, we obtain

$$
\begin{aligned}
y_1\left(\frac{dz_1}{dt}\right)^2 - z_1\left(\frac{dy_1}{dt}\right)^2 &= - m(by_1 - az_1) + m(by_1 - az_1) \\
&\quad - \frac{m_1(by_1 - az_1)}{r_1^3} + \frac{m_2(by_1 - az_1)}{r_2^3};
\end{aligned}
\tag{12}
$$

and therefore, since $by_1 - az_1 = 0$, we have

$$
y_1 \frac{d^2 z_1}{dt^2} - z_1 \frac{d^2 y_1}{dt^2} = 0.
\tag{13}
$$

Integrating this equation, we have

$$
y_1 \frac{dz_1}{dt} - z_1 \frac{dy_1}{dt} = \alpha,
\tag{14}
$$

where α is the constant of integration. This integral equation shows that the areas described by y, as the meridian plane rotates about the x-axis, are proportional to the times.

By the aid of (1) we find

$$\left(\frac{dy_1}{dt}\right)^2 + \left(\frac{dz_1}{dt}\right)^2 = \left(\frac{dy}{dt}\right)^2 + y^2\left(\frac{d\phi}{dt}\right)^2, \tag{15}$$

$$y_1\frac{dz_1}{dt} - z_1\frac{dy_1}{dt} = y^2\frac{d\phi}{dt}. \tag{16}$$

Substituting these values in (11) and (14), we have

$$\left. \begin{aligned} \tfrac{1}{2}\left\{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2\right\} &= -\tfrac{1}{2}m\,r_1^2 - \tfrac{1}{2}m\,r_2^2 - \tfrac{1}{2}m_3 r_3^2 - \tfrac{1}{2}m'\,r_4^2 - \tfrac{1}{2}m'\,r_5^2 \\ &+ \frac{m_1}{r_1} + \frac{m_2}{r_2} + \frac{m_4 + im_5}{r_4} + \frac{m_4 - im_5}{r_5} + \tfrac{1}{2}\cdot\frac{\beta}{x^2} - \tfrac{1}{2}\cdot\frac{\alpha^2}{y^2} + h, \end{aligned} \right\} \tag{17}$$

$$y^2\frac{d\phi}{dt} = \alpha, \tag{18}$$

in which the variables y_1, z_1, a, b no longer appear.

Differentiating the equation $y = \sqrt{y_1^2 + z_1^2}$ twice with regard to t, we get

$$\frac{d^2y}{dt^2} = \frac{z_1\dfrac{d^2z_1}{dt^2} + y_1\dfrac{d^2y_1}{dt^2}}{y} + \frac{(y_1^2 + z_1^2)\left\{\left(\dfrac{dy_1}{dt}\right)^2 + \left(\dfrac{dz_1}{dt}\right)^2\right\} - \left(y_1\dfrac{dy_1}{dt} + z_1\dfrac{dz_1}{dt}\right)^2}{y^3}. \tag{19}$$

The second term of the right-hand member of this equation reduces to

$$\frac{1}{y^3}\left(y_1\frac{dz_1}{dt} - z_1\frac{dy_1}{dt}\right)^2 = \frac{\alpha^2}{y^3},$$

and the first term becomes

$$\begin{aligned} &-m(y+c) - m(y-c) - m_3 y - m'y - m'y \\ &\quad - \frac{m_1(y+c)}{r_1^3} - \frac{m_2(y-c)}{r_2^3} - \frac{(m_4+im_5)y}{r_4^3} - \frac{(m_4-im_5)y}{r_5^3}. \end{aligned}$$

Substituting these values in (19), we find that the original equations of motion may be replaced by the following system:

$$\left. \begin{aligned} \frac{d^2x}{dt^2} &= -mx - mx - m_3 x - m'(x+ic) - m'(x-ic) \\ &\quad - \frac{m_1 x}{r_1^3} - \frac{m_2 x}{r_2^3} - \frac{(m_4+im_5)(x+ic)}{r_4^3} - \frac{(m_4-im_5)(x-ic)}{r_5^3} - \frac{\beta}{x^3}, \\ \frac{d^2y}{dt^2} &= -m(y+c) - m(y-c) - m_3 y - m'y - m'y \\ &\quad - \frac{m_1(y+c)}{r_1^3} - \frac{m_2(y-c)}{r_2^3} - \frac{(m_4+im_5)y}{r_4^3} - \frac{(m_4-im_5)y}{r_5^3} + \frac{\alpha^2}{y^3}, \\ \frac{d\phi}{dt} &= \frac{\alpha}{y^2}. \end{aligned} \right\} \tag{20}$$

The first two of these equations involve only the variables x and y and define the position of the moving body in the meridian plane. The third equation gives the motion of rotation about the x-axis.

If we set

$$U = -\tfrac{1}{2} m r_1^2 - \tfrac{1}{2} m r_2^2 - \tfrac{1}{2} m_3 r_3^2 - \tfrac{1}{2} m' r_4^2 - \tfrac{1}{2} m' r_5^2$$

$$+ \frac{m_1}{r_1} + \frac{m_2}{r_2} + \frac{(m_4 + i m_5)}{r_4} + \frac{(m_4 - i m_5)}{r_5} + \tfrac{1}{2} \frac{\beta}{x^2} - \tfrac{1}{2} \frac{\alpha^2}{y^2},$$

the equations (20) may be written in the form

$$\frac{d^2 x}{dt^2} = \frac{\partial U}{\partial x}, \quad \frac{d^2 y}{dt^2} = \frac{\partial U}{\partial y}, \quad \frac{d\phi}{dt} = -\frac{\partial U}{\partial \alpha}.$$

On introducing elliptic coordinates λ, μ defined by the equations

$$\lambda = \tfrac{1}{2}(r_1 + r_2), \qquad \mu = \tfrac{1}{2}(r_1 - r_2), \tag{21}$$

the functions T and U become

$$T = \tfrac{1}{2}(\lambda^2 - \mu^2)\left\{ \frac{1}{\lambda^2 - c^2}\left(\frac{d\lambda}{dt}\right)^2 - \frac{1}{\mu^2 - c^2}\left(\frac{d\mu}{dt}\right)^2 \right\},$$

$$U = \frac{1}{\lambda^2 - \mu^2}\left\{ \left(-(m + \tfrac{1}{2} m_3 + m')\lambda^4 + c^2(\tfrac{1}{2} m_3 + 2 m')\lambda^2 + (m_1 + m_2)\lambda \right.\right.$$

$$+ 2 m_4 \sqrt{\lambda^2 - c^2} + \tfrac{1}{2}\frac{\beta c^2}{\lambda^2 - c^2} - \tfrac{1}{2}\frac{\alpha^2 c^2}{\lambda^2}\left. \right) + \left((m + \tfrac{1}{2} m_3 + m')\mu^4 \right.$$

$$\left.\left. - c^2(\tfrac{1}{2} m_3 + 2 m')\mu^2 - (m_1 - m_2)\mu - 2 m_5 \sqrt{c^2 - \mu^2} - \tfrac{1}{2}\frac{\beta c^2}{\mu^2 - c^2} + \tfrac{1}{2}\frac{\alpha^2 c^2}{\mu^2} \right) \right\},$$

by virtue of the relations :

$$r_1 = \lambda + \mu, \qquad r_2 = \lambda - \mu, \qquad r_3 = \sqrt{\lambda^2 + \mu^2 - c^2},$$

$$r_4 = \sqrt{\lambda^2 - c^2} + \sqrt{\mu^2 - c^2}, \quad r_5 = \sqrt{\lambda^2 - c^2} - \sqrt{\mu^2 - c^2},$$

$$c^2 y^2 = \lambda^2 \mu^2, \quad c^2 x^2 = (\lambda^2 - c^2)(c^2 - \mu^2).$$

Both T and U have the form required for the separation of the variables in the Hamilton-Jacobi equation. U, however, is an irrational function of the coordinates λ and μ, but it is real for all real values of $\lambda > c$ and for all values of μ numerically less than c. It is a real function, therefore, for all values of λ and μ consistent with their definition.

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Setting $H = T - U$, the differential equations defining the motion in the meridian plane assume the canonical form

$$\begin{cases} \dfrac{d\lambda}{dt} = \dfrac{\partial H}{\partial p_1}, & \dfrac{dp_1}{dt} = -\dfrac{\partial H}{\partial \lambda}, \\[2ex] \dfrac{d\mu}{dt} = \dfrac{\partial H}{\partial p_2}, & \dfrac{dp_2}{dt} = -\dfrac{\partial H}{\partial \mu}, \end{cases}$$

where the conjugate variables p_1 and p_2 are given by the relations

$$p_1 = \frac{\partial T}{\partial \lambda'}, \quad p_2 = \frac{\partial T}{\partial \mu'}.$$

The Hamilton-Jacobi equation,

$$\tfrac{1}{2}\left\{ \frac{\lambda^2 - c^2}{\lambda^2 - \mu^2}\left(\frac{\partial W}{\partial \lambda}\right)^2 - \frac{\mu^2 - c^2}{\lambda^2 - \mu^2}\left(\frac{\partial W}{\partial \mu}\right)^2 \right\} = U + h,$$

becomes, on inserting the value of U in full and arranging,

$$\tfrac{1}{2}(\lambda^2 - c^2)\left(\frac{\partial W}{\partial \lambda}\right)^2 - \left\{ -(m + \tfrac{1}{2}m_3 + m')\lambda^4 + (\tfrac{1}{2}m_3 c^2 + 2m'c^2 + h)\lambda^2 \right.$$

$$\left. + (m_1 + m_2)\lambda + 2m_4\sqrt{\lambda^2 - c^2} + \tfrac{1}{2}\frac{\beta c^2}{\lambda^2 - c^2} - \tfrac{1}{2}\frac{\alpha^2 c^2}{\lambda^2} \right\}$$

$$= \tfrac{1}{2}(\mu^2 - c^2)\left(\frac{\partial W}{\partial \mu}\right)^2 + \left\{ (m + \tfrac{1}{2}m_3 + m')\mu^4 - (\tfrac{1}{2}m_3 c^2 + 2m'c^2 + h)\mu^2 \right.$$

$$\left. - (m_1 - m_2)\mu - 2m_5\sqrt{c^2 - \mu^2} - \tfrac{1}{2}\frac{\beta c^2}{\mu^2 - c^2} + \tfrac{1}{2}\frac{\alpha^2 c^2}{\mu^2} \right\}.$$

In this equation the variables are separated. Hence by setting each member equal to the same constant and putting for short

$$\Lambda = f(\lambda) + (m_1 + m_2)\lambda^5 + 2m_4\lambda^4\sqrt{\lambda^2 - c^2} - 2m_4 c^2\lambda^2\sqrt{\lambda^2 - c^2}$$
$$- (m_1 + m_2)c^2\lambda^3,$$

$$M = f(\mu) + (m_1 - m_2)\mu^5 - 2m_5\mu^4\sqrt{c^2 - \mu^2} + 2m_5 c^2\mu^2\sqrt{c^2 - \mu^2}$$
$$- (m_1 - m_2)c^2\mu^3,$$

$$f(s) = -(m + \tfrac{1}{2}m_3 + m')s^8 + (mc^2 + m_3 c^2 + 3m'c^2 + h)s^6$$
$$- (\tfrac{1}{2}m_3 c^4 + 2m'c^4 + hc^2 - k)s^4 - (k + \tfrac{1}{2}\alpha^2 - \tfrac{1}{2}\beta)c^2 s^2 + \tfrac{1}{2}\alpha^2 c^4,$$

we obtain the two equations

$$\frac{\partial W}{\partial \lambda} = \frac{\sqrt{2}}{\lambda(\lambda^2 - c^2)}\sqrt{\Lambda}, \quad \frac{\partial W}{\partial \mu} = \frac{\sqrt{2}}{\mu(\mu^2 - c^2)}\sqrt{M} \tag{22}$$

to determine the function W.

Multiplying equations (22) by $d\lambda$, $d\mu$ respectively and integrating, we find

$$W(\lambda) = \sqrt{2} \int \frac{d\lambda}{\lambda(\lambda^2 - c^2)} \sqrt{\Lambda}, \quad W(\mu) = \sqrt{2} \int \frac{d\mu}{\mu(\mu^2 - c^2)} \sqrt{M},$$

$$W = W(\lambda) + W(\mu).$$

Two integrals of the equations of motion then are

$$t + h_1 = \frac{1}{\sqrt{2}} \int \frac{\lambda^3 d\lambda}{\sqrt{\Lambda}} + \frac{1}{\sqrt{2}} \int \frac{\mu^3 d\mu}{\sqrt{M}}, \qquad (23)$$

$$k_1 = \frac{1}{\sqrt{2}} \int \frac{\lambda \, d\lambda}{\sqrt{\Lambda}} + \frac{1}{\sqrt{2}} \int \frac{\mu \, d\mu}{\sqrt{M}} \qquad (24)$$

the first of which defines the time corresponding to any position of the moving body in its path and the second the path described in the meridian plane.

The intermediary integrals (22) define the velocities, and they may be written as follows:

$$(\lambda^2 - \mu^2) \frac{d\lambda}{dt} = \frac{\sqrt{2}}{\lambda} \sqrt{\Lambda}, \quad (\lambda^2 - \mu^2) \frac{d\mu}{dt} = -\frac{\sqrt{2}}{\mu} \sqrt{M}. \qquad (25)$$

From (23) and (24), we find by differentiation

$$\sqrt{2} \, dt = \frac{\lambda^3 d\lambda}{\sqrt{\Lambda}} + \frac{\mu^3 d\mu}{\sqrt{M}}, \qquad (26)$$

$$\frac{\lambda \, d\lambda}{\sqrt{\Lambda}} + \frac{\mu \, d\mu}{\sqrt{M}} = 0, \qquad (27)$$

and with the aid of (26) and (27) we derive from the third of the equations (20) the relation

$$d\phi = -\frac{a c^2}{\sqrt{2}} \left(\frac{d\lambda}{\lambda \sqrt{\Lambda}} + \frac{d\mu}{\mu \sqrt{M}} \right), \qquad (28)$$

and after integration

$$\phi + a_1 = -\frac{a c^2}{\sqrt{2}} \left\{ \int \frac{d\lambda}{\lambda \sqrt{\Lambda}} + \int \frac{d\mu}{\mu \sqrt{M}} \right\}, \qquad (29)$$

which defines ϕ, the angle of rotation of the meridian plane. With this integral the solution of the problem by quadratures is complete. The three final integrals are (23), (24), (29), and the six arbitrary constants of integration are h, k, h_1, k_1, a, a_1, their values depending upon the initial conditions of the motion.

By their definition (21), the coordinates λ, μ must always satisfy the inequalities

$$\infty > \lambda \geq c; \qquad - c \geq \mu \geq c,$$

and in order that there may be real motion, they must also satisfy the conditions

$$\Lambda \geq 0, \qquad M \geq 0,$$

not only at the beginning of motion but also throughout the motion. Further, λ can never take the value $+ c$ nor approach $+ c$ indefinitely, for in this case the potential function U becomes infinite. For the same reason, μ can never take the values $\pm c$ nor zero, nor approach these values indefinitely. Therefore λ must have a minimum value, say a_1, greater than $+ c$, and μ must have a
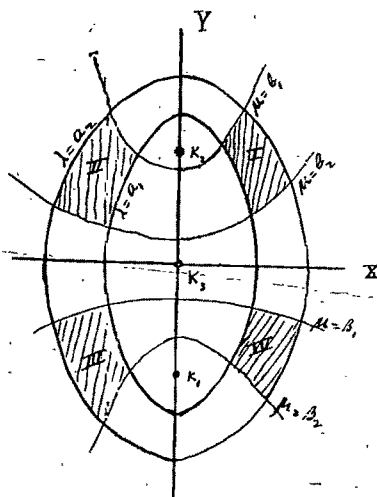
FIG. 2.

maximum and a minimum value in each of the intervals $(0, + c)$, $(- c, 0)$. Let the maximum and minimum values in the interval $(0, + c)$ be respectively b_1 and b_2, and in the interval $(- c, 0)$, β_1 and β_2 respectively. Again, if we suppose all the masses to be essentially positive, then for increasing values of λ, the function Λ finally becomes negative and remains negative. Hence λ can not increase indefinitely and the maximum value of λ for which Λ is positive or zero we shall denote by a_2. Hence λ and μ must satisfy the following conditions:

$$a_2 \geq \lambda \geq a_1, \qquad b_1 \geq \mu \geq b_2, \qquad \beta_1 \geq \mu \geq \beta_2.$$

The path described by the moving body M in the meridian plane, therefore, must always remain within one of the four trapezoidal regions of this plane bounded by ellipses $\lambda = a_1$, $\lambda = a_2$ and by the hyperbolas $\mu = b_1$, $\mu = b_2$, or by the hyperbolas $\mu = \beta_1$, $\mu = \beta_2$, as indicated in Fig. 2.

If, at the beginning of motion, M is in one of these four regions, say I, it must always remain within this region. As the meridian plane describes the angle ϕ, the region I will generate a tube-like region in space and within this space the motion of M must take place. Whatever the nature of the trajectory within this space may be, it can never extend beyond it.

If $b_1 = \beta_2$, $b_2 = \beta_1$ numerically, then as the meridian plane rotates about its axis, the spaces generated by the regions I and II will be the same as those generated by IV and III respectively, but if $b_1 \neq \beta_2$, $b_2 \neq \beta_1$ numerically, this will not be the case. It may happen, however, that the spaces generated by I and IV, or II and III, overlap; then, supposing M to be in I at the beginning of motion, the trajectory is limited by the combined spaces generated by I and IV after a semi-revolution of the meridian plane.

Let a_0 and b_0 be multiple roots of the equations $\Lambda = 0$ and $M = 0$ respectively. Then, if at the beginning and throughout the motion we have $\lambda = a_0$, the motion will be upon an arc of the ellipse $\lambda = a_0$; and, as the meridian plane rotates about the x-axis, this ellipse will generate an ellipsoid, upon a certain portion of which the body M must move. Likewise, if we have $\mu = b_0$ at the beginning and throughout the motion, the body M will move upon an arc of the hyperbola $\mu = b_0$ in the meridian plane, and in space, upon a certain portion of the hyperboloid of revolution generated by the hyperbola branch $\mu = b_0$. If we have at the same time $\lambda = a_0$ and $\mu = b_0$, no motion will be possible in the meridian plane and the orbit in space will be a circle whose axis is the x-axis.

A further discussion of the conditions for the kinds of motion just mentioned will be given later.

When the motion is supposed to take place in a fixed plane passing through the five centres of force, that is, when $\alpha = 0$, we have

$$U = \frac{1}{\lambda^2 - \mu^2} \left\{ \left(- (m + \tfrac{1}{2} m_3 + m') \lambda^4 + (\tfrac{1}{2} m_3 + 2 m') c^2 \lambda^2 + (m_1 + m_2) \lambda \right. \right.$$
$$\left. + 2 m_4 \sqrt{\lambda^2 - c^2} - \tfrac{1}{2} \cdot \frac{\beta c^2}{\lambda^2 - c^2} \right) - \left(- (m + \tfrac{1}{2} m_3 + m') \mu^4 + (\tfrac{1}{2} m_3 + 2 m') c^2 \mu^2 \right.$$
$$\left. \left. + (m_1 - m_2) \mu - 2 m_5 \sqrt{c^2 - \mu^2} - \tfrac{1}{2} \cdot \frac{\beta c^2}{\mu^2 - c^2} \right) \right\},$$

$$\Lambda = f(\lambda) + (m_1 + m_2) \lambda^3 + 2 m_4 \sqrt{\lambda^2 - c^2},$$

$$M = f(\mu) + (m_1 - m_2) \mu^3 - 2 m_5 \sqrt{c^2 - \mu^2},$$

$$f(s) = - (m + \tfrac{1}{2} m_3 + m') s^6 + (m c^2 + m_3 c^2 + 2 m' c^2 + h) s^4 + k s^2 + \tfrac{1}{2} \beta c^2 - k c^2.$$

When μ becomes zero, U becomes infinite. When either λ or μ becomes numerically equal to c, U remains finite, but again becomes infinite when $\lambda = \mu = c$; that is, when there is a collision of M with the real focus, K_1, or K_2. Hence the trajectory may cross the y-axis but not the x-axis, and it can not pass through the real foci K_1, K_2.

· For sufficiently large values of λ, the function Λ becomes and remains negative as λ continues to increase. If we represent the greatest value of λ for which Λ changes from positive to negative values by l, then λ must satisfy the inequality

$$+ c \leq \lambda \leq l$$

in all cases of real motion.
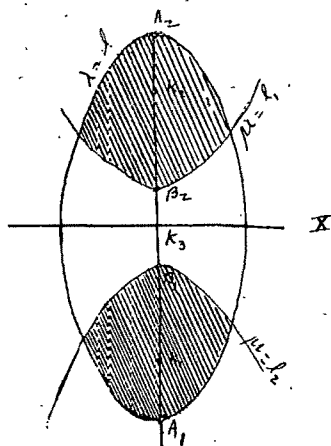
Fig. 3.

Since U becomes infinite when μ becomes zero, μ can not approach zero indefinitely. Representing the lower limit of the positive values and the upper limit of the negative values which μ may take by l_1 and l_2 respectively, μ must always satisfy the inequalities

$$+ c \geq \mu \geq l_1, \qquad l_2 \geq \mu \geq - c$$

in all cases of real motion.

From these limitations on λ and μ, it follows that the trajectory must lie wholly within one or the other of the two domains of the fixed plane, bounded by the ellipse $\lambda = l$ and the hyperbola branches $\mu = l_1$, $\mu = l_2$, as in Fig. 3.

For $\lambda = c$ the ellipse degenerates into the straight line segment $K_1 K_2$; for $\mu = c$ the hyperbola degenerates into the straight line $K_1 K_2$ beyond K_2, and for

$\mu = -c$ into the straight line $K_1 K_2$ beyond K_1. Hence, if at the beginning and throughout the motion we have $\lambda = c$, the body M must move on the line $K_1 K_2$, either on the segment $B_2 K_2$ or the segment $B_1 K_1$, not including the limiting positions K_2, K_1. On the other hand, if at the beginning and throughout the motion we have $\mu = +c$, the body M must move upon the line segment $K_2 A_2$, excluding K_2; and similarly for $\mu = -c$, the motion takes place on the straight line segment $K_1 A_1$, excluding K_1.

If we have at the beginning and throughout the motion $\lambda = a_0$, where a_0 is a multiple root of $\Lambda = 0$, the motion will be upon an arc of the ellipse $\lambda = a_0$. Likewise, if we have $\mu = b_0$ at the beginning and throughout the motion, the motion will be upon an arc of the hyperbola $\mu = b_0$.

For a fixed system of values of the masses involved, the existence of multiple roots of the equations $\Lambda = 0$, $M = 0$ depends upon the arbitrary constants h and k that is, upon the initial conditions of the motion. From this it is evident that the same conic which can be described by all the forces acting at the same time can also be described when any one of the forces R_1, R_2, R_3 acts alone or when the two forces R_4, R_5 alone act. This fact Lagrange considered most remarkable and called attention to it in his work on the problem of two fixed centres in the *Mécanique Analytique.* Legendre[*] also noticed it. The above result and the theorems of Lagrange and Legendre are corollaries of the following general theorem due to Bonnet:[†]

- Si plusieurs masses m, m', m'', ..., respectivement soumises à l'action des forces F, F', F'', ..., et partant toutes d'un point A avec des vitesses v_0, v_0', v_0'', ..., de grandeur différente mais de même direction, décrivent la même courbe ACB, la masse quelconque M, soumise à l'action de la résultante des forces F, F', F'', ..., et partant du point A avec une vitesse V_0 ayant la même direction que les vitesses v_0, v_0', v_0'', ..., décrira encore la courbe ACB, pourvu que les forces F, F', F'', ..., indépendantes du temps et que la force vive initiale MV de la masse M soit égale à la somme

$$m v_0^2 + m' v_0'^2 + m'' v_0''^2 + \ldots,$$

des forces vives initiales des masses m, m', m'', ...

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[*] *Traité de fonctions elliptiques,* Vol. I, p. 426.

[†] *Journ. de Math.,* Vol. IX, pp. 113–115; *Œuvres de Lagrange,* Vol. XII, Note IV, pp. 353–355.

We shall now consider the motion of a particular case of the preceding problem by the method employed by Charlier[*] in the discussion of the motion of the original problem of two fixed centres when the motion is in a plane through the line of centres. The problem is as follows:

To determine the motion of a free body M in space attracted toward two fixed centres K_1, K_2 by forces $R_1 = - m r_1 - \dfrac{m_1}{r_1^2}$, $R_2 = - m r_2 - \dfrac{m_2}{r_2^2}$ respectively, and by a third force $R_3 = - \dfrac{\beta}{x^3}$ parallel to the line $K_1 K_2$.
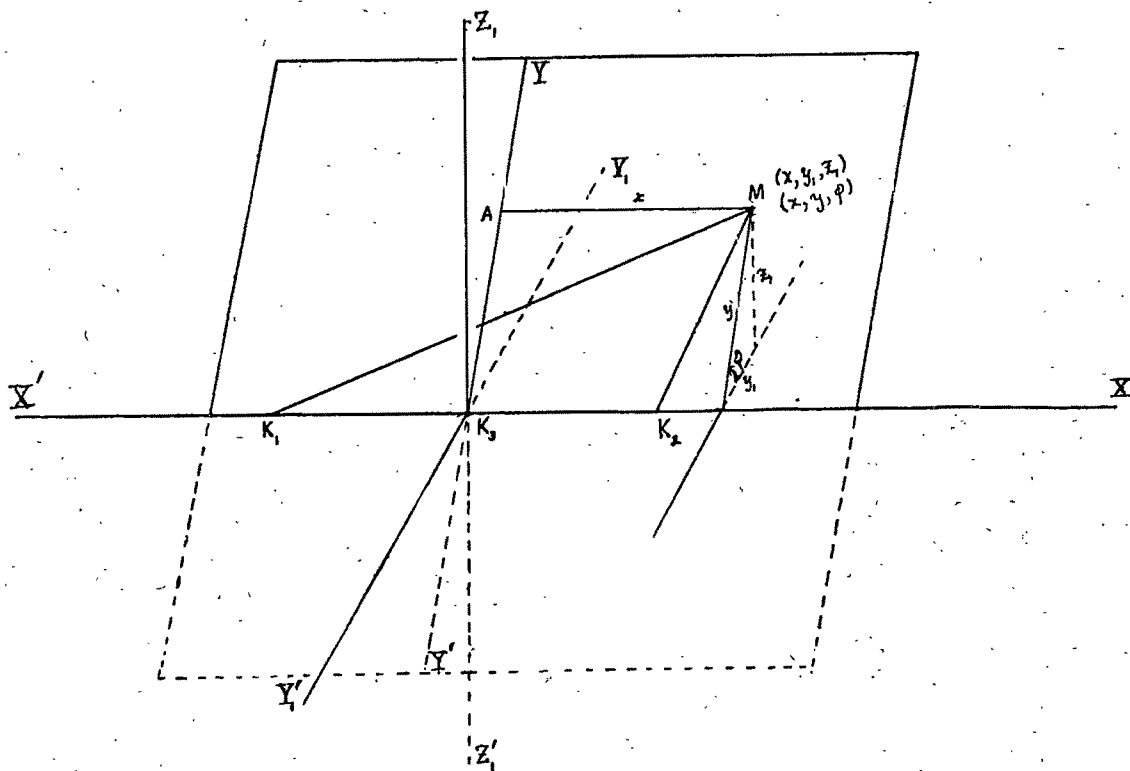
In this case K_1, K_2 are fixed. The plane $K_1 M K_2$ is supposed to rotate about the line $K_1 K_2$ as axis, coinciding with the fixed plane $X K_3 Y_1$ at the beginning of motion and describing the angle ϕ in the time t, as indicated in Fig. 4. AM, or x, is the distance of the moving body M from the fixed plane $Y_1 K_3 Z_1$. Hence the third force, R_3, varies inversely as the cube of the distance of M from the fixed plane $Y_1 K_3 Z_1$.

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[*] *Mechanik des Himmels*, Vol. I, pp. 117–163.

If we take the mass M as the unit mass, the equations of motion are

$$\frac{d^2 x}{dt^2} = -m(x+c) - m(x-c) - \frac{m_1(x+c)}{r_1^3} - \frac{m_2(x-c)}{r_2^3} - \frac{\beta}{x^3},$$

$$\frac{d^2 y_1}{dt^2} = -m y_1 - m y_1 - \frac{m_1 y_1}{r_1^3} - \frac{m_2 y_1}{r_2^3},$$

$$\frac{d z_1}{dt^2} = -m z_1 - m z_1 - \frac{m_1 z_1}{r_1^3} - \frac{m_2 y_2}{r_2^3}.$$

The further details of the solution are altogether analogous to those of the general problem already solved, so that we content ourselves with the results

$$T = \tfrac{1}{2}(\lambda^2 - \mu^2)\left\{\frac{\lambda'^2}{\lambda^2 - c^2} - \frac{\mu'^2}{\mu^2 - c^2}\right\},$$

$$U = \frac{1}{\lambda^2 - \mu^2}\left\{\left(-m\lambda^4 + (m_1 + m_2)\lambda - \frac{\tfrac{1}{2}\alpha^2 c^2}{\lambda^2 - c^2} - \frac{\tfrac{1}{2}\beta c^2}{\lambda^2}\right) - \left(-m\mu^4 + (m_1 - m_2)\mu - \frac{\tfrac{1}{2}\alpha^2 c^2}{\mu^2 - c^2} - \frac{\tfrac{1}{2}\beta c^2}{\mu^2}\right)\right\},$$

$$\frac{d\phi}{dt} = \frac{\alpha}{y^2},$$

$$W = \int \frac{d\lambda}{\lambda(\lambda^2 - c^2)}\sqrt{\Lambda} + \int \frac{d\mu}{\mu(\mu^2 - c^2)}\sqrt{M},$$

$$\lambda(\lambda^2 - \mu^2)\frac{d\lambda}{dt} = \sqrt{\Lambda}, \tag{30}$$

$$\mu(\lambda^2 - \mu^2)\frac{d\mu}{dt} = -\sqrt{M}, \tag{31}$$

$$dt = \frac{\lambda^3 d\lambda}{\sqrt{\Lambda}} + \frac{\mu^\varepsilon d\mu}{\sqrt{M}}, \tag{32}$$

$$\frac{\lambda d\lambda}{\sqrt{\Lambda}} + \frac{\mu d\mu}{\sqrt{M}} = 0, \tag{33}$$

$$d\phi = -\alpha c^2\left(\frac{\lambda d\lambda}{(\lambda^2 - c^2)\sqrt{\Lambda}} + \frac{\mu d\mu}{(\mu^2 - c^2)\sqrt{M}}\right), \tag{34}$$

$$t + h_1 = \int \frac{\lambda^3 d\lambda}{\sqrt{\Lambda}} + \int \frac{\mu^3 d\mu}{\sqrt{M}}, \tag{35}$$

$$h_1 = \int \frac{\lambda d\lambda}{\sqrt{\Lambda}} + \int \frac{\mu d\mu}{\sqrt{M}}, \tag{36}$$

$$\phi + \alpha_1 = -\alpha c^2\left(\int \frac{\lambda d\lambda}{(\lambda^2 - c^2)\sqrt{\Lambda}} + \int \frac{\mu d\mu}{(\mu^2 - c^2)\sqrt{M}}\right). \tag{37}$$

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The integration constants are h, k, α, h_1, k_1, α_1. The symbols Λ and M stand for the following expressions:

$$\Lambda = -2m\lambda^8 + 2(mc^2 + h)\lambda^6 + 2(m_1 + m_2)\lambda^5 - 2(hc^2 - k)\lambda^4$$
$$-2(m_1 + m_2)c^2\lambda^3 - (2k + \alpha^2 + \beta)c^2\lambda^2 + \beta c^4,$$
$$M = -2m\mu^8 + 2(mc^2 + h)\mu^6 + 2(m_1 - m_2)\mu^5 - 2(hc^2 - k)\mu^4$$
$$-2(m_1 - m_2)c^2\mu^3 - (2k + \alpha^2 + \beta)c^2\mu^2 + \beta c^4;$$

or, setting for short

$$A = 2m, \quad B = 2mc^2 + h, \quad C_1 = 2(m_1 + m_2), \quad C_2 = 2(m_1 - m_2),$$
$$D = 2(hc^2 - k), \quad E_1 = 2(m_1 + m_2)c^2, \quad E_2 = 2(m_1 - m_2)c_i^2,$$
$$F = 2k + \alpha^2 + \beta, \quad G = \beta c^4:$$
$$\Lambda = -A\lambda^8 + B\lambda^6 + C_1\lambda^5 - D\lambda^4 - E_1\lambda^3 - F\lambda^2 + G,$$
$$M = -A\mu^8 + B\mu^6 + C_2\mu^5 - D\mu^4 - E_2\mu^3 - F\mu^2 + G.$$

It was shown by Serret[*] that under certain conditions.

$$\Lambda M = 0$$

is a singular solution of the problem of Euler or of that of Lagrange, and as his reasoning is directly applicable to the above problem we may conclude that this equation is also a singular solution of the problem in hand. In the original problem of Euler Λ and M are of the fourth degree in λ and μ respectively, in Lagrange's problem they are of the sixth, and in this problem they are of the eighth.

The condition for such singular solution is that $\lambda - \lambda_0$, λ_0 being the initial value of λ, be a multiple factor of Λ of order m, $m \geq 2$; or similarly, that $\mu - \mu_0$, μ_0 being the initial value of μ, be a multiple factor of M of order n, $n \geq 2$; that is, λ_0 must be at least a double root of the equation $\Lambda = 0$; or μ_0 at least a double root of $M = 0$.

The trajectory in the meridian plane will therefore be the ellipse $\lambda = \lambda_0$, or the hyperbola $\mu = \mu_0$, according as

$$\Lambda_0 = 0, \quad \left(\frac{\partial \Lambda}{\partial \lambda}\right)_0 = 0,$$

or

$$M_0 = 0, \quad \left(\frac{\partial M}{\partial \mu}\right)_0 = 0,$$

where Λ_0 and $\left(\frac{\partial \Lambda}{\partial \lambda}\right)_0$ are the values of Λ and its derivative with regard to λ

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when $\lambda = \lambda_0$, and likewise M_0 and $\left(\dfrac{\partial M}{\partial \mu}\right)_0$ are the values of M and its derivative with regard to μ when $\mu = \mu_0$.

As the meridian plane rotates about the line of centres, the ellipse $\lambda = \lambda_0$ will generate an ellipsoid of revolution and the hyperbola $\mu = \mu_0$ will generate an hyperboloid of revolution. Upon one of these surfaces the motion takes place.

But if $\lambda = \lambda_0$ and $\mu = \mu_0$ at the same time where λ_0 and μ_0 occur at least twice as roots of the equations $\Lambda = 0$ and $M = 0$ respectively, then no motion is possible in the meridian plane, for in this case

$$\frac{d\lambda}{dt} = 0, \qquad \frac{d\mu}{dt} = 0;$$

in space, however, the trajectory will be a circle whose axis is the axis of revolution and whose centre is K_3. This kind of motion was first remarked by Legendre,[*] and later by Andrade.[†]

With regard to equations of the form

$$\left(\frac{dx}{dt}\right)^2 = F(x).$$

Charlier[‡] established certain theorems which are immediately available for our discussion. The equation is supposed to be one capable of interpretation in a mechanical problem, and therefore x as well as its first derivative with regard to t may be regarded as real, finite, and continuous for all finite values of t.

In the first place let a be any real root whatever of $F(x) = 0$ of order m, m being a positive integer. Let $x = x_0$ at the beginning of motion and let x_0 be so taken that no zero value of $F(x)$ lies between a and x_0. During the motion let x approach a continuously by increasing or decreasing values according as x_0 is less than or greater than a. Under these assumptions, Charlier succeeded in establishing the following results:

First, for $m \geq 2$, x can not reach the value a in a finite interval of time, and being real and continuous, it can not therefore pass the value a.

Second, for $m = 1$, x will reach the value a within a finite time; but it can not pass this value, for then the derivative will change sign, which is not admissible.

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[*] *Traité de fonctions elliptiques*, Vol. II, p. 529.

[†] *Journal d'École Polytechnique*, Cahier 60, 1890.

[‡] *Mechanik des Himmels*, Vol. I, pp. 117–163.

In other words, x can never pass a zero value of $F(x)$.

In the second place, let a and b be two consecutive roots of $F(x) = 0$ of orders m and n respectively, let $a < b$ and let x_0 lie between a and b at the beginning of motion. Then it follows immediately from the above theorem that if $x = x_0$ at the beginning of motion, it must always remain within the interval (a, b). If $m = n = 1$ and if x starts from x_0 toward a, it will continue to decrease until it reaches the value a; then it turns and continues to increase until it takes the value b, when it changes its direction again toward a, and so on indefinitely. The motion in this case is one of oscillation, or what Charlier calls *libration motion*. The limits a and b are called the *libration limits*.

If $m \geq 2$ and $n = 1$, and if x starts from x_0, x_0 being in the interval (a, b) at the beginning of motion, toward a, it will continue to decrease but will never reach the value a; it approaches a as a limit and can not attain this limit in a finite interval of time. If x starts from x_0 toward b, it will reach b in a finite interval of time; it will then change its direction and continue to decrease toward a as before. If $m \geq 2$ and $n \geq 2$, then, in whichever direction x moves at the beginning of motion, it will continue to move indefinitely, approaching either a or b as a limit. Charlier calls this kind of motion *limitation motion* and the limits, *limitation limits*.

These theorems will now be applied to equations (30) and (31).

The coordinates λ and μ have been introduced by the equations

$$\lambda = \tfrac{1}{2}(r_1 + r_2), \qquad \mu = \tfrac{1}{2}(r_1 - r_2),$$

where λ and μ are the parameters of a system of confocal conics having their real foci at the fixed centres of force, K_1, K_2. If we denote the distance $K_1 K_2$ by $2c$, then the least value of λ is $+c$, while the greatest and least values of μ are $+c$ and $-c$ respectively. Since λ is the semi-major axis of the ellipse, the semi-minor axis is $\sqrt{\lambda^2 - c^2}$; of the hyperbola the semi-transverse axis is μ, and the semi-conjugate axis is $\sqrt{c^2 - \mu^2}$. Hence the equations of the ellipses and hyperbolas are respectively

$$\frac{x^2}{\lambda^2} + \frac{y^2}{\lambda^2 - c^2} = 1, \qquad \frac{x^2}{\mu^2} - \frac{y^2}{c^2 - \mu^2} = 1,$$

and therefore

$$c^2 x^2 = \lambda^2 \mu^2, \qquad c^2 y^2 = (\lambda^2 - c^2)(c^2 - \mu^2).$$

From these defining equations of λ and μ, it follows that λ and μ must satisfy the inequalities

$$\infty > \lambda > c, \qquad + c \geq \mu \geq -c.$$

In order that the quantities under the radicals in the integrals may be positive, that is, in order that there may be real motion, λ and μ must also satisfy the conditions

$$\Lambda \geq 0, \qquad M \geq 0,$$

respectively. All of the above conditions λ and μ must satisfy throughout the motion.

The potential function U can be written

$$U = \frac{1}{2(\lambda^2 - \mu^2)} \left\{ \left(\frac{\Lambda}{\lambda^2(\lambda^2 - c^2)} - 2(h\lambda^2 + k) \right) - \left(\frac{M}{\mu^2(\mu^2 - c^2)} - 2(h\mu^2 + k) \right) \right\}.$$

It is evident that, of the possible values λ and μ, U becomes infinite for $\lambda = c$ and for $\mu = \pm c$ or zero. Therefore λ and μ can not be allowed to take these values nor to approach them indefinitely, if we wish to avoid infinite discontinuities in U. Hence there must be a lower limit to the possible values of λ, as $l > c$; likewise, an upper and a lower limit to the possible positive values of μ which we denote by l_1 and l_1' respectively, and an upper and lower limit to the possible negative values of μ which will be denoted by l_2' and l_2 respectively. The following inequalities must therefore be fulfilled by λ and μ respectively:

$$\infty > \lambda \geq l, \qquad l_1 \geq \mu \geq l_1', \qquad l_2' \geq \mu \geq l_2.$$

For any value of $\lambda > c$ for which Λ vanishes, the function U will be finite, however little this value of λ may differ from c. Hence the least value of the possible values of λ will always be less than or at most equal to the least value of λ for which Λ vanishes. In other words, all the positive roots of the equation $\Lambda = 0$ which are greater than $+c$ are also greater than or equal to l. Similarly, all the positive values of μ in the interval $(0, +c)$ for which M vanishes lie in the interval (l_1', l_1), and all the negative values of μ in the interval $(-c, 0)$ for which M vanishes lie in the interval (l_2, l_2').

When μ becomes zero, not only U but also $d\mu/dt$ becomes infinite. But for this value of μ, no discontinuities occur in the three final integrals of the solution; that is, the time t as well as the trajectory in the meridian plane and in space

is finite and continuous, as one can easily determine. Hence in a discussion of the trajectories no further attention need be given to these discontinuities in U and $d\mu/dt$ and it will be sufficient that λ and μ satisfy the conditions

$$\infty > \lambda \geq l, \qquad l_2 \geq \mu \geq l_1.$$

Without loss of generality we may take $m_1 > m_2$. Then, since c^2 and a^2 are always positive, if we regard m, m_1, m_2 and β as essentially positive, no change of sign can take place in any of the coefficients of Λ and M which are independent of the arbitrary constants h and k. Hence A, C_1, C_2, E_1, E_2, G are all positive, while B, D, F may be positive or negative, depending for their sign upon the values of h and k.

When $\lambda = 0$, the expression Λ is positive, and when $\lambda = \pm c$, Λ is negative. Λ must therefore have one real root between 0 and $+c$, and one between $-c$ and 0. By the aid of Descartes' Rule of Signs, we find also that Λ can not have more than three positive, and it can not have more than five negative roots. $\Lambda = 0$ must therefore have at least one positive root, but may have three; and it must have at least one negative root, but may have three or five. The same is true relative to the roots of the equation $M = 0$.

Further, for a certain value of λ, the function Λ becomes negative and remains negative for all greater values of λ. Hence there must be a greatest value of λ for which Λ is positive or zero; that is, there must be a greatest value for which Λ changes from positive to negative. This value, which we shall denote by a_1, must be a root of $\Lambda = 0$. Let us suppose $a_1 > c$. Since Λ is negative for $\lambda = +c$ and positive for values of λ less than a_1, Λ must also change sign for some value of λ between $+c$ and a_1; there must be a positive root of $\Lambda = 0$ in the interval $(+c, a_1)$. It is evident then, that Λ is positive for all values of λ in the interval (a_2, a_1) and for no other values of λ greater than $+c$. Since a_1 and a_2 are both greater than l, λ may take all values in the interval (a_2, a_1) including the limits a_2 and a_1.

We have seen that one positive root of $\Lambda = 0$ lies between 0 and $+c$. If a_1 be this root, then there can be but one positive real root; if a_1 be different from this root but less than $+c$, then a_2 will also be less than $+c$. In neither of these cases is motion possible. In all cases of real motion, therefore, we must have both a_1 and a_2 greater than $+c$, and λ must satisfy the inequality $a_1 \geq \lambda \geq a_2$.

As to the roots of $M = 0$ we have remarked that those only can have any influence upon the motion which satisfy the inequality $l_2 \geq \mu \geq l_1$, and that $M = 0$ must have one positive root but may have three, and one negative root but may have three or five. We denote the positive roots by b_1, b_2, b_3 and the negative roots by β_1, β_2, β_3, β_4, β_5. By b_1 we shall always mean the single positive root, or, if there be three positive roots, the greatest of the positive roots in the interval $(0, l_1)$ for which M changes from positive to negative values. Similarly, by β_1 we mean the negative root which must always exist, or, if there be more than one negative root, the greatest numerically of the negative roots in the interval $(l_2, 0)$ which causes M to change from negative to positive values for increasing values of μ.
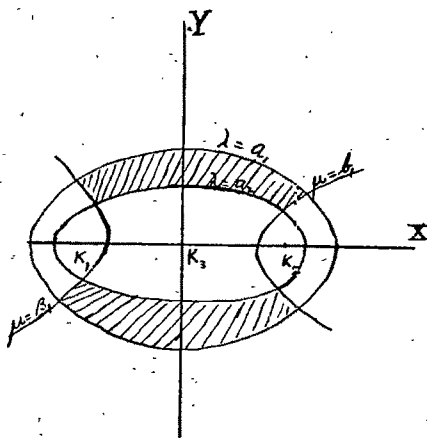
Fig. 5.

Hence it is clear that the trajectory of the moving body M in the meridian plane must remain wholly within one of two regions enclosed by $\lambda = a_1$, $\lambda = a_2$ and by the hyperbola branches $\mu = \beta_1$, $\mu = b_1$, as shown in Fig. 5. As the meridian plane rotates about the line of centres, these regions will generate a tube-like portion of space within which the trajectory must always remain.

Considering the definitions and restrictions with regard to the values of μ for which M vanishes as determined above, it is found that there are 105 different combinations of the roots of the equation $M = 0$ which must be taken into account in a complete discussion of the motion, and each of these combinations must be associated with each of the two combinations of the roots of the equation $\Lambda = 0$, which are

$$A_1: \quad a_1 > a_2, \qquad A_2: \quad a_1 = a_2,$$

thus making in all 210 different combinations of the roots in question, which may influence the motion.

The positive roots of $M = 0$ may occur in any one of the following ways:

B_1: b_2, b_3 imaginary, or real and greater than l_1,

B_2: b_2, b_3 real, unequal and less than b_1,

B_3: b_2, b_3 real, equal and less than b_1,

B_4: b_2, b_3 real, equal, greater than b_1 and less than l_1,

B_5: b_2, b_3 real and equal to b_1.

The ways in which the negative roots may occur relative to β_1 and l_1 are accounted for in the following manner:

First, any two of them, as β_2, β_3:

C_1: β_2, β_3 imaginary, or real and less than l_2,

C_2: β_2, β_3 real, unequal and greater than β_1,

C_3: β_2, β_3 real, equal and greater than β_1,

C_4: β_2, β_3 real, equal, less than β_1 and greater than l_2,

C_5: β_2, β_3 real and equal to β_1.

Second, the remaining two, β_4, β_5:

D_1: β_4, β_5 imaginary, or real and less than l_2,

D_2: β_4, β_5 real, unequal and greater than β_1,

D_3: β_4, β_5 real, equal and greater than β_1,

D_4: β_4, β_5 real, equal, less than β_1 and greater than l_2,

D_5: β_4, β_5 real and equal to β_1.

We must also take certain of the C's with certain of the D's in order to obtain relations between the negative roots which are not given above. These are:

$C_3 D_{21}$: β_4, β_5 real, unequal, greater than β_1, and $\beta_2 = \beta_3 > \beta_5 > \beta_4$,

$C_3 D_{22}$: β_4, β_5 real, unequal, greater than β_1, and $\beta_5 > \beta_2 = \beta_3 > \beta_4$,

$C_3 D_{23}$: β_4, β_5 real, unequal, greater than β_1, and $\beta_5 > \beta_4 > \beta_3 = \beta_2 > \beta_1$,

$C_3 D_{24}$: β_4, β_5 real, unequal, greater than β_1, and $\beta_5 = \beta_3 = \beta_2 > \beta_4$,

$C_3 D_{25}$: β_4, β_5 real, unequal, greater than β_1, and $\beta_5 > \beta_4 = \beta_3 = \beta_2$,

$C_3 D_{31}$: β_4, β_5 real, equal, greater than β_1, and $\beta_2 = \beta_3 = \beta_4 = \beta_5$,

$C_3 D_{32}$: β_4, β_5 real, equal, greater than β_1, and $\beta_5 = \beta_4 > \beta_3 = \beta_2 > \beta_1$,

$C_4 D_{41}$: β_4, β_5 real, equal, less than β_1, greater than l_2 and $\beta_5 = \beta_4 = \beta_3 = \beta_2$,

$C_4 D_{42}$: β_4, β_5 real, equal, less than β_1, greater than l_2 and $\beta_2 = \beta_3 > \beta_4 = \beta_5$.

All the combinations of the roots of $\Lambda = 0$ and $M = 0$, which are actually different and which can have any influence on the motion, can now be formed. They are given by the following formulæ:

$$P_{ijk1} = A_i B_j C_k D_1, \qquad P_{ij3(2,k)} = A_i B_j C_3 D_{2,k},$$
$$P_{ijm2} = A_i B_j C_m D_2, \qquad P_{ijp(p,l)} = A_i B_j C_p D_{p,l},$$
$$P_{ijn3} = A_i B_j C_n D_3, \qquad P_{ij5n} = A_i B_j C_5 D_n,$$

where

$$i = 1, 2; \qquad j = 1, 2, 3, 4, 5; \qquad k = 1, 2, 3, 4, 5;$$
$$l = 1, 2; \qquad m = 2, 4, 5; \qquad n = 4, 5; \qquad p = 3, 4.$$

These different combinations, 210 in all, give rise to numerous possibilities of motion, depending upon the initial values of λ and μ; but, as one can easily determine, all the possible cases of motion so arising will fall into one of the following six general classes:

    I.   Libration in λ and libration in μ,

    II.   Libration in λ and limitation in μ,

    III.   Libration in λ and $\mu = $ constant,

    IV. $\lambda = $ constant and libration in μ,

    V. $\lambda = $ constant and limitation in μ,

    VI. $\lambda = $ constant and $\mu = $ constant.

To give a clear idea of the nature of the motion in the various possible cases it will be sufficient to discuss a few of them so as to illustrate each of the above six general classes. To this end we shall consider further only the combinations $P_{133(2,1)}$ and $P_{233(2,1)}$.

In the first of these combinations we have

$$a_1 > a_2, \qquad b_3 = b_2 < b_1, \qquad \beta_2 = \beta_3 > \beta_5 > \beta_4,$$

and therefore the following possibilities of motion are presented:

    (1) $a_1 > \lambda_0 > a_2$: $b_1 > \mu_0 > b_2 = b_3$;

    (2) $a_1 > \lambda_0 > a_2$: $\mu_0 = b_2 = b_3$;

    (3) $a_1 > \lambda_0 > a_2$: $b_2 = b_3 > \mu_0 > \beta_2 = \beta_3$;

    (4) $a_1 > \lambda_0 > a_2$: $\mu_0 = \beta_2 = \beta_3$;

    (5) $a_1 > \lambda_0 > a_2$: $\beta_2 = \beta_3 > \mu_0 > \beta_5$;

    (6) $a_1 > \lambda_0 > a_2$: $\beta_4 > \mu_0 > \beta_1$,

λ_0 and μ_0 being the initial values of λ and μ respectively. In each of the six

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cases there is libration in λ. In (6) there is also libration in μ and therefore (6) belongs to class I. In (1), (3), (5) there is limitation in μ as in class II, and (2) and (4) belong to class III.

If at the beginning of the motion the conditions (6) are satisfied, the trajectory of the moving point in the meridian plane must always remain within one of the two trapezoidal regions bounded by the ellipses $\lambda = a_1$, $\lambda = a_2$ and by the hyperbola branches $\mu = \beta_4$, $\mu = \beta_1$. Since there is libration in both λ and μ, these variables will take the values a_1, a_2 and β_4, β_1 respectively an indefinite number of times and the trajectory will meet the bounding arcs an indefinite number of times. As the meridian plane rotates about its axis, this region of the meridian plane will generate a tube-like portion of space, bounded by the surfaces of the ellipsoids of revolution $\lambda = a_1$, $\lambda = a_2$ and the hyperboloids of revolution $\mu = \beta_4$, $\mu = \beta_1$. Within this tube the trajectory must always remain, meeting the limiting surfaces an indefinite number of times. The trajectory will be a coil-like curve.

If conditions (1) are satisfied at the beginning of motion, the trajectory in the meridian plane must always remain within the region bounded by the ellipses $\lambda = a_1$, $\lambda = a_2$ and the hyperbolas $\mu = b_1$, $\mu = b_2$. The variable λ will take the values a_1, a_2 an indefinite number of times while μ may take the value b_1 once, if at all, and must then approach b_2 as a limit, not reaching it in a finite interval of time. Hence the trajectory will meet the bounding arcs of the ellipses an indefinite number of times and approach the bounding arc of the hyperbola $\mu = b_2$ in an asymptotic manner. In space the trajectory will be a wave-like, spiral-like curve meeting the surfaces of the ellipsoids of revolution $\lambda = a_1$, $\lambda = a_2$ an indefinite number of times while it continues to approach asymptotically the surface of the hyperboloid of revolution $\mu = b_2$, which is intercepted between the ellipsoids $\lambda = a_1$, $\lambda = a_2$.

When conditions (3) or (5) hold at the beginning of motion, the trajectory will be of the same general character as that corresponding to conditions (1).

When (2) and (4) are satisfied at the beginning of motion, we have libration in λ and $\mu = $ constant throughout the motion. In the meridian plane the trajectory consists of the arc of the hyperbola intercepted between the ellipses $\lambda = a_1$, $\lambda = a_2$ and the motion is one of oscillation upon this arc. In space the trajectory must remain upon the surface of the hyperboloid $\mu = b_2$ intercepted between the ellipsoids generated by $\lambda = a_1$, $\lambda = a_2$. It is therefore a wave-like curve crossing this surface and meeting the bounding circles an indefinite number of times.

In the second combination selected for further discussion, $P_{288(2,1)}$, we have

$$a_1 = a_2, \qquad b_3 = b_2 < b_1, \qquad \beta_2 = \beta_3 > \beta_5 > \beta_4 > \beta_1,$$

and therefore we have the following possibilities depending upon the initial values λ_0 and μ_0 of λ and μ respectively:

(1) $\lambda_0 = a_1 = a_2:$ $b_1 > \mu_0 > b_2 = b_3;$

(2) $\lambda_0 = a_1 = a_2:$ $\mu_0 = b_2 = b_3;$

(3) $\lambda_0 = a_1 = a_2:$ $b_2 = b_3 > \mu_0 > \beta_2 = \beta_3;$

(4) $\lambda_0 = a_1 = a_2:$ $\mu_0 = \beta_2 = \beta_3;$

(5) $\lambda_0 = a_1 = a_2:$ $\beta_2 = \beta_3 > \mu_0 > \beta_5;$

(6) $\lambda_0 = a_1 = a_2:$ $\beta_5 > \mu_0 > \beta_4.$

Of these different cases (6) belongs to class IV; (1), (3), (5) to class V and (2), (4) to class VI.

If conditions (6) are satisfied at the beginning of motion, the trajectory will consist of the arc of the ellipse $\lambda = a_1$, between the hyperbolas $\mu = \beta_5$, $\mu = \beta_4$. In space it will be upon the surface of the ellipsoid of revolution bounded by the circles of intersection of this surface with the two hyperboloids of revolution $\mu = \beta_5$, $\mu = \beta_4$. The trajectory will be a wave-like curve upon the ellipsoid crossing its surface and meeting the bounding circles an indefinite number of times.

If at the beginning of motion conditions (1) are fulfilled, the trajectory in the meridian plane will be the arc of the ellipse $\lambda = a_1$ intercepted between the hyperbolas $\mu = b_1$, $\mu = b_2$, but the motion will not be one of oscillation as in the preceding case. If the variable μ move initially in the direction of b_1, it will reach that value in a finite time, then change its direction toward b_2, which it approaches as a limit, not reaching it in a finite interval of time. As the meridian plane rotates about its axis, the motion will be upon the surface of the ellipsoid of revolution $\lambda = a_1$, between the hyperboloids of revolution $\mu = b_1$, $\mu = b_2$. The trajectory will be a curve winding about the ellipsoid and approaching asymptotically the circle of intersection of this surface with the hyperboloid $\mu = b_2$. In cases (3) and (5) the trajectory is of the same nature as that just described.

In cases (2) and (4) no motion is possible in the rotating plane, but in space the trajectory is a circle whose axis is the line $K_1 K_2$. In (2), for example, the trajectory is the circle of intersection of the ellipsoid of revolution $\lambda = a_1$ and the hyperboloid $\mu = b_2$.

By giving particular values to the masses and the arbitrary constants, interesting particular cases of motion may be discovered. If we take $\alpha = 0$ and make the necessary changes throughout, the cases of motion in a fixed plane passing through the line of centres will be obtained. Then, if we take $\lambda = c$ throughout the motion, the trajectory will consist of the straight line segment $K_1 K_2$; or, if we take $\mu = \pm c$ throughout the motion, the trajectory will consist of the segments of the line $K_1 K_2$ beyond the fixed centres K_2 and K_1 respectively.

PRINCETON, N. J.

Abstract Definitions of all the Substitution Groups whose Degrees do not exceed Seven.

By G. A. MILLER.

§1. INTRODUCTION.

Abstract group theory does not only owe its birth to the substitution groups of low degrees but many abstract group properties may be studied most advantageously by means of these substitution groups. The bond between these two kinds of groups of finite order is becoming more and more evident, and the difference between them is frequently only a matter of language. This is especially true as regards the transitive substitution groups and the abstract groups. In view of these facts it may be desirable to have a table by means of which one may readily obtain abstract definitions for all the substitution groups whose degrees do not exceed seven.

The fact that the same substitution group may frequently be defined abstractly in a large number of different ways constitutes an element of difficulty in the preparation of such a table. While we have tried to give one of the simplest definitions in each case, it must be admitted that in such matters simplicity is largely dependent upon the point of view. The definition which involves the smallest number of symbols does not always exhibit the properties of a group most clearly. Notwithstanding this obstacle it will doubtless often be helpful to have on hand at least one abstract definition of each one of the substitution groups in question. To increase the usefulness of the table we have added some details under the heading of Explanations. These are given just after the table and are especially extensive as regards the groups of the first few degrees.

As the cyclic group of order n may be represented by the n nth roots of unity, we shall define this group by the equation $s^n = 1$. The dihedral group of order $2m$ may be conveniently defined by $s_1^2 = 1$, $s_2^2 = 1$, $(s_1 s_2)^m = 1$; since it is generated by any two operators of order 2 whose product is of order m. This group is a special case of the one which may be obtained by extending any

abelian group (H) by means of an operator of order 2 which transforms every operator of H into its inverse. The order of the group (K) obtained in this way is evidently twice the order of H, and K involves only operators of order 2 in addition to those found in H. We proceed to give an abstract definition of K.

Consider the group generated by n operators of order 2 which are such that the product of any three of them is also of order 2. That is, we are considering (s_1, s_2, ..., s_n) when these generators satisfy the conditions:

$$s_1^2 = s_2^2 = s_3^2 = ... = s_n^2 = (s_\alpha s_\beta s_\gamma)^2 = 1; \quad \alpha, \beta, \gamma = 1, 2, ..., n.$$

It is easy to see that each of these operators transforms into its inverse the product of any two of them. This fact follows directly from the equations:

$$s_\gamma s_\alpha s_\beta s_\gamma = s_\gamma . s_\gamma s_\beta s_\alpha = s_\beta s_\alpha = (s_\alpha s_\beta)^{-1}.$$

Since each of the given n operators transforms the product of any two of them into its inverse, it results that the product of any two must be commutative with the product of any other two. That is, the operators which are the products of an even number of the operators s_1, s_2, ..., s_n are commutative with each other, and hence they generate an abelian group H.

The operators s_1, s_2, ..., s_n can clearly be so selected that the $n-1$ cyclic groups generated by $s_1 s_2$, $s_1 s_3$, $s_1 s_4$, ..., $s_1 s_n$ respectively are independent, and hence H may have $n-1$ invariants. Since any product involving an even number of factors from s_1, s_2, ..., s_n may be formed by means of the given $n-1$ products, it results that H can not involve more than $n-1$ invariants when these invariants are so selected that each of them is divisible by all those which follow it. Hence we have proved the theorem:

If n operators of order 2 are such that the product of any three of them is also of order 2, they generate a group which involves an abelian subgroup composed of half the operators of the group, and each of the remaining operators is of order 2 and transforms each operator of this abelian subgroup into its inverse. Moreover, if this abelian group involves no more than $n-1$ invariants, the given group of twice its order can always be generated by n such operators of order 2.

If H is any subgroup of index ρ under G, all the operators of G may be uniquely represented in either of the following two ways:

$$G = H + Hs_2 + Hs_3 + ... + Hs_\rho,$$
$$= H + s_2 H + s_3 H + ... + s_\rho H.$$

The sets Hs_α, $s_\alpha H$ ($\alpha = 2, 3, ..., \rho$) are known respectively as the right and the left co-sets of G as regards H, and the sets obtained by adding H to these

co-sets are known as the augmented right and left co-sets respectively. That is, the augmented co-sets always include the subgroup with respect to which the sets are formed. These definitions are useful for the purpose of formulating a theorem which is very fundamental in finding suitable abstract definitions of known groups. This theorem may be stated as follows: *Two necessary and sufficient conditions that the augmented right co-sets*

$$H + Hs_2 + Hs_3 + \ldots + Hs_\rho$$

constitute a group are that they include the product of any two of the operators $s_2, s_3, \ldots, s_\rho$ *and that they also include the inverse of each operator in these co-sets.*[*]

The fact that the conditions expressed in this theorem are necessary follows directly from the definition of a group. The fact that they are also sufficient may be established as follows. When these conditions are satisfied, the given right co-sets must include the following left co-sets:

$$s_2^{-1}H, \; s_3^{-1}H, \; \ldots, \; s_\rho^{-1}H.$$

Since no two operators of these co-sets are identical, the total number of operators in the given right co-sets are identical with those in these left co-sets. The product of any two operators in these right co-sets may be represented by $Hs_\alpha Hs_\beta$. As $s_\alpha H$ is contained in H or in a right co-set and hence also in the given augmented right co-sets, it results that these co-sets constitute a group. As a very special case of the given theorem we may state the fact that if we obtain only products of order 2 by multiplying all the operators of a group by a given operator this group and these products of order 2 must constitute a group whose order is twice the order of the original group.

Suppose now that $t_1, t_2, \ldots, t_\lambda$ is a set of generators of H and that $H + Hs_2 + Hs_3 + \ldots + Hs_\rho$ includes the inverses of each of the operators

$$t_{\alpha'} s_{\beta'}^{-1}; \quad \alpha' = 1, 2, \ldots, \lambda; \; \beta' = 2, 3, \ldots, \rho.$$

It is also assumed that $s_2, s_3, \ldots, s_\rho$ satisfy the condition imposed on them in the preceding theorem. Any operator in the product $Hs_\alpha Hs_\beta$ may be written in the form

$$Hs_\alpha t_1^{a_1} t_2^{a_2} \ldots t_\gamma^{a_\gamma} \ldots s_\beta.$$

By successive reduction this assumes the form Hs_γ, and hence the given augmented right co-sets constitute a group also in this case. To illustrate this

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[*] In the definition of co-sets it is implied that s_β is not contained in any of the sets $H, Hs_2, \ldots, Hs_\alpha$ $(\alpha < \beta)$; it is however not assumed that the augmented co-sets as regards a group (H) necessarily constitute a group.

corollary we shall employ it for the purpose of finding a simple abstract definition of the symmetric group of degree n as follows:

$$s_1^2 = s_2^2 = \ldots = s_{n-1}^2 = (s_j s_k)^3 = (s_i s_j s_i s_k)^2 = 1; \quad i \neq j \neq k;$$
$$i, j, k = 1, 2, \ldots, n-1.$$

That these equations define a group whose order is at least equal to that of the symmetric group of degree n results directly from the fact that they are satisfied when $s_1, s_2, \ldots, s_{n-1}$ are replaced respectively by the substitutions $a_1 a_n$, $a_2 a_n$, $a_3 a_n$, $\ldots$, $a_{n-1} a_n$.

Let $t_1, t_2, \ldots, t_{n-2}$ be the transforms as regards s_{n-1} of $s_1, s_2, \ldots, s_{n-2}$, in order. It is evident that the t's satisfy the conditions imposed on the s's, if the latter satisfy these conditions, except that $i, j, k = 1, 2, \ldots, n-2$. Hence we may assume that these t's generate the symmetric group of degree $n-1$, and if we can prove that this assumption implies that the s's generate the symmetric group of degree n our theorem is clearly established. It is therefore only necessary to prove that if $H = (t_1, t_2, \ldots, t_{n-2})$ is the symmetric group of degree $n-1$ then must

$$H + Hs_1 + Hs_2 + \ldots + Hs_{n-1}$$

be the symmetric group of degree n. That these operators constitute a group whose order is n times that of H results directly from the following equations together with the corollary noted above :

$$s_\alpha s_\beta = t_\alpha t_\beta t_\alpha \cdot s_\alpha \ (\alpha, \beta < n-1), \ s_{n-1} s_\alpha = t_\alpha s_{n-1}, \ s_\alpha s_{n-1} = t_\alpha s_\alpha ;$$
$$s_\beta t_\alpha = s_\beta s_{n-1} s_\alpha s_{n-1} = t_\alpha s_\beta \ (\beta < n-1, \ \alpha \neq \beta); \ s_\beta t_\alpha = s_\alpha s_{n-1}, \ \beta = n-1;$$
$$s_\beta t_\alpha = s_{n-1} s_\alpha, \ \alpha = \beta.$$

As the s's generate a group whose order is n times that of H, whenever these s's satisfy the given conditions it results by induction that they generate the symmetric group of degree n. It should be observed that a somewhat different general abstract definition for the symmetric group of degree n as well as a general definition of the alternating group has been given by Moore[*] and others, and that the present definition has been given mainly for the purpose of illustrating the theorem in question. From the following list it will become apparent that it is frequently possible to give much briefer definitions in special cases.

In a similar manner we can obtain an abstract definition of the alternating

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group of degree n as a special case of the given theorem. To do this we may start with the following $n - 1$ operators:

$$s_1^3 = s_2^2 = s_3^2 = \ldots = s_{n-2}^2 = (s_i s_j)^3 = (s_i s_j s_i s_k)^2 = 1; \quad i \neq j \neq k;$$
$$i, j, k = 1, 2, \ldots, n-2.$$

That these $n - 2$ operators generate a group whose order is at least as large as that of the alternating group of degree n becomes evident if we observe that $s_1, s_2, s_3, \ldots, s_{n-2}$ may be replaced by the following substitutions, in order, in the given defining relations: $a_1 a_2 a_n, \; a_1 a_2 \cdot a_3 a_n, \; a_1 a_2 \cdot a_4 a_n, \ldots, a_1 a_2 \cdot a_{n-1} a_n.$

If we transform $s_1, s_1^{-1}, s_2, s_3, \ldots, s_{n-2}$ by the last one of these operators and represent the resulting operators, in order, by $t_1, t_1^{-1}, t_2, t_3, \ldots, t_{n-3}$, it is evident that these t's satisfy the conditions imposed on the s's. Hence we have only to prove that the latter generate the alternating group of degree n if the former generate this group of degree $n-1$. That the latter have this property follows directly from the equations:

$$s_\alpha s_\beta = t_\alpha t_\beta t_\alpha s_\alpha \, (1 < \alpha, \, \beta < n - 2), \qquad s_1 s_3 = t_1^2 t_\beta t_1^2 s_1^2, \qquad s_1 s_{n-2} = t_1^2 t_{n-2} t_1^2 s_{n-2},$$
$$s_1^2 s_\beta = t_1 t_\beta t_1 s_1, \qquad s_1^2 s_{n-2} = t_1 s_1, \qquad s_\beta s_1 = t_1 t_\beta t_1 s_\beta, \qquad s_{n-2} s_1 = t_1 s_{n-2},$$
$$s_\beta s_1^2 = t_1^2 t_\beta t_1^2 s_\beta, \qquad s_{n-2} s_1^2 = t_1^2 s_{n-2}, \qquad s_{n-2} s_\beta = t_\beta s_{n-2}, \qquad s_\beta s_{n-2} = t_\beta s_\beta;$$
$$s_\beta t_\alpha = t_\alpha s_\beta \, (1 < \alpha, \, \beta < n-2; \alpha \neq \beta), \qquad\qquad\qquad s_{n-2} t_\alpha = s_\alpha s_{n-2},$$
$$s_\alpha t_\alpha = s_{n-2} s_\alpha, \qquad s_1 t_\alpha = t_\alpha s_1^2, \qquad s_1^2 t_\alpha = t_\alpha s_1, \qquad s_1 t_1 = t_1^2 s_{n-2},$$
$$s_1^2 t_1 = t_1^2 s_1, \qquad s_1 t_1^2 = t_1 s_1^2, \qquad s_1^2 t_1^2 = t_1 s_{n-2}, \qquad s_\alpha t_1 = t_1^2 s_\alpha,$$
$$s_\alpha t_1^2 = t_1 s_\alpha, \qquad s_{n-2} t_1 = t_1^2 s_1^2, \qquad s_{n-2} t_1^2 = t_1 s_1.$$

The similarity between the present abstract definition of the alternating group and the given abstract definition of the symmetric group should be noted.

The number of the different abstract groups which can be represented as substitution groups on seven or a smaller number of letters is 54.* Some of these groups can be represented in more than one way as substitution groups on seven or a smaller number of letters. In what follows we shall use only one such representation; that is, we shall consider only one of a set of simply isomorphic groups, and we shall represent this on the smallest possible number of letters. If this is done, the numbers of different abstract groups which may be represented as substitution groups of the various degrees from 2 to 7 are respectively 1, 2, 5, 7, 13, 26. The notation employed is practically the same as was used in the list published in the AMERICAN JOURNAL OF MATHEMATICS to which we have referred. The only change is the omission of the abbreviation

cyc after the parenthesis enclosing the generator of a cyclic group. This omission seems to be a step towards simplicity and uniformity in substitution notation.

§ 2. List of Groups.

Degree Two.

Order.	No.	Notation.	Abstract Definition.
2	1	(ab)	$s_1^2 = 1$

Degree Three.

3	1	(abc)	$s_1^3 = 1$
6	1	(abc) all	$s_1^2 = s_2^2 = (s_1 s_2)^3 = 1$

Degree Four.

4	1	$(ab)(cd)$	$s_1^2 = s_2^2 = (s_1 s_2)^2 = 1$
	2	$(abcd)$	$s_1^4 = 1$
8	1	$(abcd)_8$	$s_1^2 = s_2^2 = (s_1 s_2)^4 = 1$
12	1	$(abcd)$ pos	$s_1^2 = s_2^3 = (s_1 s_2)^3 = 1$
24	1	$(abcd)$ all	$s_1^2 = s_2^3 = (s_1 s_2)^4 = 1$

Degree Five.

5	1	$(abcde)$	$s_1^5 = 1$
6	1	$(abc)(de)$	$s_1^6 = 1$
10	1	$(abcde)_{10}$	$s_1^2 = s_2^2 = (s_1 s_2)^5 = 1$
12	1	(abc) all (de)	$s_1^2 = s_2^2 = (s_1 s_2)^6 = 1$
20	1	$(abcde)_{20}$	$s_1^4 = s_2^5 = s_1^3 s_2 s_1 s_2^3 = 1$
60	1	$(abcde)$ pos	$s_1^2 = s_2^3 = (s_1 s_2)^5 = 1$
120	1	$(abcde)$ all	$s_1^4 = s_2^5 = (s_1^2 s_2)^3 = (s_1 s_2^3)^2 = 1$

Degree Six.

8	1	$(ab)(cd)(ef)$	$s_1^2 = s_2^2 = s_3^2 = (s_1 s_2)^2 = (s_1 s_3)^2 = (s_2 s_3)^2 = 1$
	2	$(abcd)(ef)$	$s_1^4 = s_2^2 = s_1^3 s_2 s_1 s_2 = 1$
9	1	$(abc)(def)$	$s_1^3 = s_2^3 = s_1^2 s_2^2 s_1 s_2 = 1$
16	1	$(abcd)_8(ef)$	$s_1^2 = s_2^2 = s_3^2 = (s_1 s_2)^2 = (s_1 s_3)^2 = (s_2 s_3)^4 = 1$
18	1	(abc) all (def)	$s_1^2 = s_2^2 = s_3^3 = (s_1 s_2)^3 = s_1 s_3 s_1 s_3^2 = s_2 s_3 s_2 s_3^2 = 1$
	2	$\{(abc)$ all (def) all$\}$ pos	$s_1^2 = s_2^2 = s_6^2 = (s_1 s_2 s_3)^2 = (s_1 s_2)^3 = (s_1 s_3)^3 = 1$
24	1	$(abcd)$ pos (ef)	$s_1^2 = s_2^3 = (s_1 s_2^2 s_1 s_2)^2 = 1$
36	1	(abc) all (def) all	$s_1^6 = s_2^6 = (s_1 s_2)^2 = s_1^5 s_2^2 s_1 s_2^2 = s_2^5 s_1^2 s_2 s_1^2 = 1$
	2	$(abcdef)_{36}$	$s_1^4 = s_2^4 = (s_1 s_2)^2 = (s_1^2 s_2^2)^3 = (s_1 s_2^3)^3 = 1$
48	1	$(abcd)$ all (ef)	$s_1^2 = s_2^6 = (s_1 s_2^2)^4 = (s_1 s_2^3)^2 = 1$
72	1	$(abcdef)_{72}$	$s_1^2 = s_2^4 = (s_1 s_2)^5 = (s_1 s_2^3 s_1 s_2)^2 = 1$

360	1	(abcdef) pos	$s_1^3 = s_2^4 = (s_1 s_2^2)^4 = (s_1 s_2^3 s_1^2 s_2)^2 = 1$
720	1	(abcdef) all	$s_1^6 = s_2^4 = (s_1^2 s_2^2)^4 = (s_1^2 s_2^3 s_1^4 s_2)^2$
			$= s_1^5 s_2^3 s_1 (s_1^2 s_2)^2 s_2 (s_1^2 s_2)^3 s_2 = 1$

Degree Seven.

7	1	(abcdefg)	$s_1^7 = 1$
10	1	(abcde)(fg)	$s_1^{10} = 1$
12	1	(ab)(cd)(efg)	$s_1^2 = s_2^6 = s_1 s_2^5 s_1 s_2 = 1$
	2	(abcd)(efg)	$s_1^{12} = 1$
	3	{(abcd)(efg) all} pos	$s_1^4 = s_2^3 = s_1^3 s_2 s_1 s_2 = 1$
14	1	(abcdefg)$_{14}$	$s_1^2 = s_2^2 = (s_1 s_2)^7 = 1$
20	1	(abcde)$_{10}$(fg)	$s_1^2 = s_2^2 = (s_1 s_2)^{10} = 1$
21	1	(abcdefg)$_{21}$	$s_1^3 = s_2^7 = s_1^2 s_2 s_1 s_2^5 = 1$
24	1	(ab)(cd)(efg) all	$s_1^2 = s_2^2 = s_3^2 = (s_1 s_2)^6 = (s_1 s_3)^2 = (s_2 s_3)^2 = 1$
	2	(abcd)(efg) all	$s_1^2 = s_2^{12} = s_1 s_2 s_1 s_2^7 = 1$
	3	(abcd)$_8$(efg)	$s_1^2 = s_2^{12} = s_1 s_2 s_1 s_2^5 = 1$
	4	{(abcd)$_8$ com (efg) all} dim	$s_1^4 = s_2^6 = s_1^2 s_2 s_1^2 s_2^5 = s_1^3 s_2 s_1 s_2 s_1^2 = 1$
	5	{(abcd)$_8$ cyc (efg) all} dim	$s_1^2 = s_2^2 = (s_1 s_2)^{12} = 1$
36	1	(abcd) pos (efg)	$s_1^6 = s_2^3 = (s_1^3 s_2)^3 = s_1^4 s_2 s_1^2 s_2^3 = 1$
40	1	(abcde)$_{20}$(fg)	$s_1^4 = s_2^{10} = s_1^3 s_2 s_1 s_2^3 = 1$
42	1	(abcdefg)$_{42}$	$s_1^6 = s_2^7 = s_1^5 s_2 s_1 s_2^4 = 1$
48	1	(abcd)$_8$(efg) all	$s_1^2 = s_2^2 = s_3^{12} = (s_1 s_2)^2 = s_1 s_3 s_1 s_3^7 = s_2 s_3 s_2 s_3^5 = 1$
72	1	(abcd) all (efg)	$s_1^3 = s_2^{12} = (s_1 s_2^3)^2 = s_2^8 s_1^2 s_2^4 s_1 = 1$
	2	(abcd) pos (efg) all	$s_1^6 = s_2^6 = (s_1^2 s_2^3)^3 = (s_1^3 s_2^2)^2 = (s_1^3 s_2^3)^2 = s_1^2 s_2^2 s_1^4 s_2^4 = 1$
	3	{(abcd) all (efg) all} pos	$s_1^4 = s_2^2 = s_3^3 = (s_1 s_2)^3 = (s_2 s_3)^2 = s_1^3 s_3 s_1 s_3 = 1$
120	1	(abcde) pos (ef)	$s_1^2 = s_2^6 = (s_1 s_2^2)^5 = (s_1 s_2^3)^2 = 1$
144	1	(abcd) all (efg) all	$s_1^6 = s_2^{12} = (s_1^2 s_2^3)^2 = (s_1^3 s_2^4)^2 = s_1^2 s_2^4 s_1^4 s_2^8 = s_1^3 s_2^3 s_1^3 s_2^9 = 1$
168	1	(abcdefg)$_{168}$	$s_1^7 = s_2^3 = (s_1 s_2)^2 = (s_2 s_1^5)^4 = 1$
240	1	(abcde) all (fg)	$s_1^4 = s_2^{10} = (s_1^2 s_2^2)^3 = (s_1 s_2^6)^2 = s_1^3 s_2^5 s_1 s_2^5 = 1$
2520	1	(abcdefg) pos	$s_1^3 = s_2^2 = s_3^2 = s_4^2 = s_5^2 = (s_i s_j)^3 = (s_i s_j s_i s_k)^2 = 1;$
			$i \neq j \neq k; \; i, j, k = 1, 2, \ldots, 5$
5040	1	(abcdefg) all	$s_i^2 = (s_j s_k)^3 = (s_i s_j s_i s_k)^2 = 1, \; i \neq j \neq k,$
			$i, j, k = 1, 2, \ldots, 6$

§ 3. EXPLANATIONS.

Groups of Degrees Three and Four.

The two groups of degree 3 are respectively cyclic and dihedral, while those of degree 4 are cyclic, dihedral, or of genus zero. Hence all of these groups belong to well-known categories of abstract groups. The symmetric group of

degree 3 is the smallest non-abelian dihedral group. It may also be defined as the only non-abelian group involving only two complete sets of conjugates besides the identity, as the holomorph of the group of order 3, as the only non-abelian group that involves only three operators of order 2 and is generated by them, as the only non-abelian group that has the symmetric group of order 6 for its group of isomorphisms,[*] as the group of movements of the equilateral triangle, as the group generated by subtracting from a given number x_1 and by dividing x_1^2, or by multiplying by 2 and by adding unity, if we start with a given parameter and reduce the results modulo 3, as the only non-abelian group in which the number of complete sets of conjugate operators (excluding the identity) is exactly equal to the number of distinct prime factors in the order of the group.[†]

The non-cyclic group of order 4 is known under a number of different names. Among these are the following: the quadratic group, the anharmonic ratio group, the rectangle group, the axial group, the four-group. It is the non-cyclic group of lowest order, and it presents itself in various parts of mathematics as these different names imply. For instance, in the ordinary complex plane it is the group generated by the operations of inversion and reflection on the x-axis; on a line it is the group of the permutation of four points such that the anharmonic ratio of these points is left unchanged; in number theory it is the group formed by multiplying the numbers which are prime to 8 and reducing them and their products modulo 3; etc. the cyclic group of order 4 is perhaps best known as the one which is generated by $\sqrt{-1}$.

The substitution group of degree 4 and of order 8 is dihedral and is commonly called the *octic* group. It is the holomorph of the cyclic group of order 4 as well as the group of movements of a square. It is the only group of order 2^m which contains exactly five operators of order 2 and is generated by them. It is also generated by the two operations of starting with a given angle and taking the supplement and the complement as well as by the two operations of subtracting from x_1 and of dividing $x_1^2/2$. The alternating group of degree 4 is commonly known as the *tetrahedral* group, since it is the group of movements of the regular tetrahedron. It does not contain any subgroup of order 6 and is the group of lowest order that does not contain a subgroup whose order is an arbitrary divisor of the order of the group. The symmetric group of degree 4 is

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[*] *Transactions of the American Mathematical Society*, Vol. I (1900), p. 399.

[†] *Ibid.*, Vol. V (1904), p. 505.

called the *octahedral* group, since it is the group of movements of the regular octahedron. The following are some of its characteristic properties: It is the holomorph of the axial group; it is the group of movements of the cube; it contains exactly nine operators of order 2 which generate it and are such that the order of the product of at least two of them is 4, while no two of them have a product whose order exceeds 4; it is the smallest group that has a non-abelian commutator subgroup and is generated by two operators of orders 2 and 3 respectively having a commutator of order 3;[*] it is generated by three cyclic subgroups of order 4 which are non-invariant and do not involve a common subgroup of order 2 nor generate other cyclic subgroups of order 4.[†]

Groups of Degrees Five and Six.

As all the substitution groups of degree 5, with the exception of the alternating and the symmetric group, are cyclic, dihedral or metacyclic, their properties may have been sufficiently explained in what precedes. Moreover, the icosahedral group is so well known that it seems unnecessary to enter into a discussion of its properties. The given abstract definition of the symmetric group may, however, deserve a word of explanation. It is clear that s_1^2, s_2 satisfy the ordinary conditions of generators of the icosahedral group and that this group is transformed into itself by s_1, since the last relation implies that $s_1^{-1}s_2^2s_1 = s_2^3s_1^2$. It therefore results that (s_1, s_2) is of order 120. Since s_1, s_2 may be replaced respectively by the substitutions *aceb* and *abcde*, it must be the symmetric group of this order.

The given abstract definition of the substitution group of degree 6 and order 24 is in accord with the theorem,[‡] *If an operator of order 2 and an operator of order 3 have a commutator of order 2, they generate either the tetrahedral group or the direct product of this group and a group of order 2.* To verify the given abstract definition of the second group of order 36 we may proceed as follows: The two conjugate operators $s_1^2s_2^2$, $s_1s_2^2s_1$ are commutative, since their commutator is $(s_2^2s_1)^4 = 1$. Hence they generate a group of order 9. This is evidently invariant under (s_1, s_2) and each of its operators is transformed into its inverse by each of the two operators s_1^2, s_2^2. The group of order 18 generated by s_1^2, $s_1^2s_2^2$, $s_1s_2^2s_1$ is also invariant under (s_1, s_2), since $s_2^2s_1^2s_2 = s_1s_2^2s_1 \cdot s_2^2$. Hence it results that $(s_1, s_1^2s_2^2, s_1s_2^2s_1)$ is of order 36. It must include s_2, since $(s_1s_2^3)^3 = 1 = s_1s_2^2s_1^3s_2^2s_1s_2^3$. If we omit the condition $(s_1s_2^3)^3 = 1$, the remaining conditions define the direct product of this group of order 36 and the group of order 2.

[*] *Transactions of the American Mathematical Society*, Vol. IX (1908), p. 76.

[†] *Mathematische Annalen*, Vol. LXIV (1907), p. 344.

[‡] *Transactions of the American Mathematical Society*, Vol. IX (1908), p. 68.

The given abstract definition of the group of order 48 results immediately from the fact that it is the direct product of the octahedral group and the group of order 2. The group of order 72 is evidently generated by the substitutions ab and $afcd . be$, and these two substitutions satisfy the given abstract definitions. It is therefore only necessary to prove that if s_1, s_2 satisfy these defining relations they can not generate a group whose order exceeds 72. Let $t_1 = (s_1 s_2)^2$, and observe that t_1 and $s_1 t_1 s_1$ are commutative. The group of order 9 $(t_1, s_1 t_1 s_1)$ is evidently invariant under (s_1, s_2). It is also easy to verify that the subgroup of order 18 $(t_1, s_1 t_1 s_1, s_1 s_2^3 s_1 s_2)$ is invariant, and that this is identical with $(t_1, s_1 t_1 s_1, s_2^2)$. As a final step we may observe that $(t_1, s_1 t_1 s_1, s_2^2, s_1)$ is an invariant subgroup of order 36, since $s_2^3 s_1 s_2 = s_2^3 s_1 s_2 s_1 . s_1$. The given abstract definition for the alternating group of degree 6 is the one discussed by Dickson in the *Bulletin of the American Mathematical Society*, Vol. IX (1903), p. 303. The abstract definition of the symmetric group of degree 6 can be directly obtained from this definition of the alternating group.

Groups of Degree Seven.

The given abstract definition of the group of order 36 results from the ordinary definition of the tetrahedral group and the fact that this group of order 36 is the direct product of the tetrahedral group and the group of order 3. The definition of the group of order 48 may be illustrated by means of the substitutions $s_1 = de, s_2 = ac, s_3 = abcd . efg$. The definition of the first group of order 72 results directly from that of the octahedral group, while that of the second becomes evident if we observe that (s_1^2, s_2^3) is the tetrahedral group and that (s_1^3, s_2^2) is the symmetric group of order 6. In the definition of the third group of order 72 it may be observed that (s_1, s_3) is the octahedral group, in accord with the common definition of this group. The definition of the group of order 120 results immediately from that of the icosahedral group.

The given abstract definition of the group of order 144 may be verified by observing that (s_1^2, s_2^3) is the octahedral group, while (s_1^3, s_2^4) is the symmetric group of order 6, and each operator of one of these groups is commutative with every operator of the other. The defining relations of the simple group of order 168 agree with those given by Dyck in the *Mathematische Annalen*, Vol. XX (1882), p. 41. The definition of the group of order 240 results directly from that of the symmetric group of degree 5, while the definitions of the alternating and the symmetric group are illustrations of the general definitions of these groups to which we referred in the Introduction.

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The Attraction of a Homogeneous Spherical Segment.

By G. Greenhill.

Mr. G. W. Hill has arrived at the unexpected result that the attraction of the homogeneous segment of a sphere (a flat lens) can be made to depend on the complete elliptic integral, first, second, and third, and thence is expressible by the function $F\phi$ and $E\phi$ of Legendre's Table IX (AMERICAN JOURNAL OF MATHEMATICS, Vol. XXIX, No. 4).

The result is unexpected, as the ordinary method of dissection leads to an integral of intractable nature, where elements of the integral are symmetrical with respect to the axis CO of the segment in Fig. 1.

But by cutting the segment up into slices by planes perpendicular to CP, drawn from C, the centre of the sphere, to the attracted point P, each slice QRQ' (Fig. 2) is the segment of a circle of which CP is the axis (the line through the centre of a circle perpendicular to its plane) and the components of attraction perpendicular and parallel to CP, as well as the potential of the segment at P, are given by simple functions.

This is the dissection employed by Mr. G. W. Hill; and a final integration by parts enables him to reduce the components of attraction of the spherical segment to an algebraical quadrature, which proves to be of elliptic character.

The object of this memoir is to resume the consideration of this elliptic integral and to show that the result can be made to depend on the complete elliptic integral of the first and second kind, and on two complete integrals of the third kind, expressible by incomplete integrals of the first and second kind.

Interpreted geometrically these two third elliptic integrals can be taken to represent the apparent area, or magnetic potential, of the base AQB (Fig. 3) of the segment, as seen from P, and another point P' on the radius CP which is inverse to P with respect to the spherical surface.
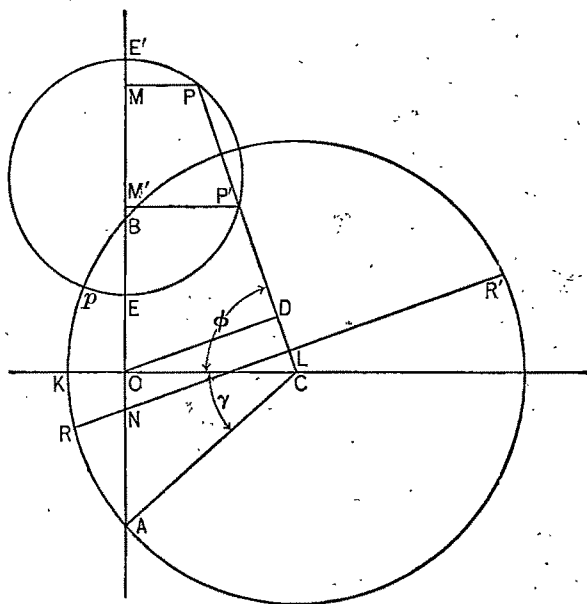
FIG. 1.
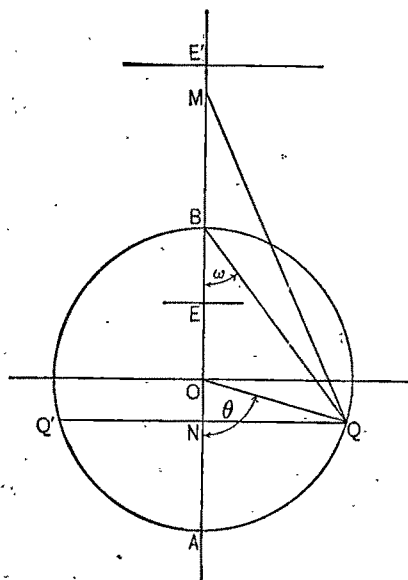
FIG. 3.
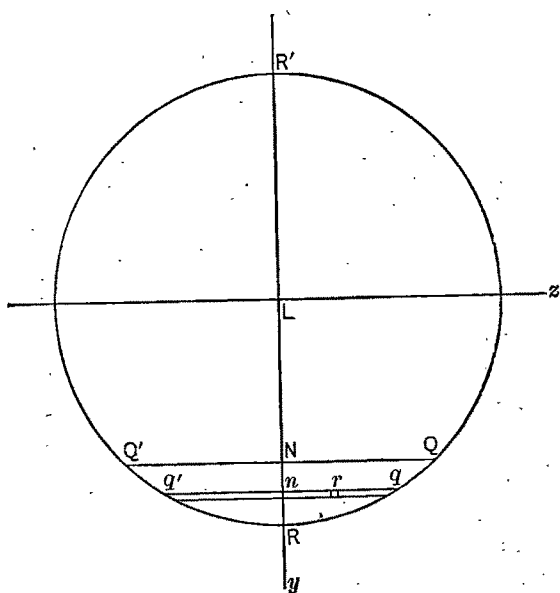
FIG. 2.
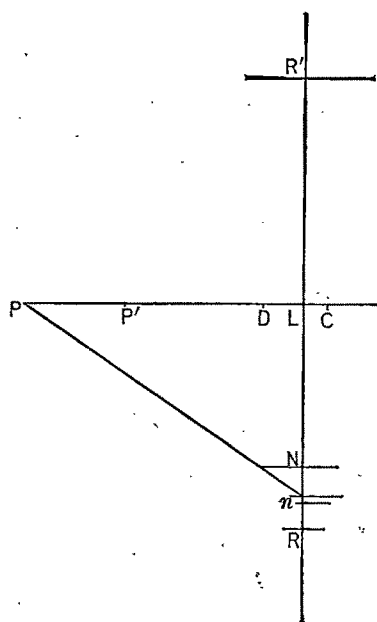
FIG. 4.

The analytical reduction is thus similar to that employed in the "Elliptic Integral in Electromagnetic Theory," *Transactions American Mathematical Society*, October, 1907, referred to in the sequel as *Trans. A. M. S.*; and to facilitate the identification of results the same notation will be employed, taken from Maxwell's *E. and M. (Electricity and Magnetism).*

Used in conjunction with Mr. G. W. Hill's notation, this will require a change of his a into c to represent the radius of the sphere, retaining Maxwell's a to represent the radius of the base of the segment.

As we shall not require Mr. Hill's b, no change need be made to avoid confusion with Maxwell's b, representing MP, the distance of P from the plane of the base AB.

The other symbols, x, γ, ϕ, are retained, as employed by Mr. Hill; but we have found it convenient to change his x' into r, although we shall have occasion to use r to denote PQ.

There are so many quantities to be distinguished that the same letter must be used occasionally, with due warning, for more than one meaning, if the conventional notation is to be preserved.

Following Mr. Hill, we begin by the calculation of Y, the component of the attraction perpendicular to CP, by slicing the spherical segment into segments of a circle on the same axis CP, made by planes perpendicular to CP; the component attraction of the circular segment perpendicular to CP then represents

$$\frac{1}{G\rho}\frac{dY}{dx}$$

in the spherical segment, ρ denoting the density and G the constant of gravitation.

Attraction of the Circular Segment QRQ' at a Point P on its Axis CP
(Fig. 2 and 4).

1. The attraction at P in the direction Pn of the line element qq' is:

$$\int_{-nq}^{nq}\frac{Pn}{Pr}\cdot\frac{dz}{Pr^2}=\int_0^{nq}\frac{2\,Pn\,dz}{(Pn^2+z^2)^{\frac{3}{2}}}=\frac{n\,r}{Pr\,.\,Pn}\Big]_{-nq}^{nq}=\frac{2nq}{PQ\,.\,Pn},\qquad(1)$$

so that the component attraction of the segment QRQ' in the direction LR perpendicular to CP is:

$$\int_{LN}^{LR}\frac{Ln}{Pn}\cdot\frac{2\,nq\,dy}{PQ\,.\,Pn}=\frac{2}{PQ}\int\frac{\sqrt{(LQ^2-y^2)}\,y\,dy}{PL^2+y^2}$$

$$=2\,PQ\int\frac{y\,dy}{(PL^2+y^2)\sqrt{(LQ^2-y^2)}}-\frac{2}{PQ}\int\frac{y\,dy}{\sqrt{(LQ^2-y^2)}}$$

$$=2\,\mathrm{th}^{-1}\frac{NQ}{PQ}-2\,\frac{NQ}{PQ}=2\,J_1-2\,\frac{NQ}{PQ},\qquad(2)$$

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where

$$J_1 = \text{th}^{-1}\frac{NQ}{PQ} = \text{ch}^{-1}\frac{PQ}{PN} = \text{sh}^{-1}\frac{QN}{PN}, \qquad (3)$$

or

$$J_1 = \tfrac{1}{2}\log\frac{PQ+NQ}{PQ-NQ} = \tfrac{1}{2}\log\frac{1+P}{1-P}, \qquad (4)$$

in Mr. Hill's notation for P; and

$$\text{th } J_1 = \frac{NQ}{PQ} = P = \sin QPN, \qquad (5)$$

and the angle QPN is the hyperbolic amplitude of J_1.

2. The notation, representing the various lines on the figures, is taken to agree as closely as possible with Maxwell and Mr. Hill:

$$OA = a, \qquad OM = A, \qquad MP = b,$$
$$OCP = \phi, \qquad OCA = \gamma, \qquad BA = c,$$
$$OA = c\sin\gamma = a, \qquad OC = c\cos\gamma = a\cot\gamma,$$
$$CP = r \text{ (or } x'), \qquad CL = x, \qquad LP = r - x,$$
$$MP = c\cos\gamma - r\cos\phi = b, \ldots\ldots$$

Supposing x to range between

$$x_2 = a = c\cos(\phi-\gamma) \text{ and } x_3 = b = c\cos(\phi+\gamma), \qquad (1)$$

$$x_2 - x \cdot x - x_3 = -x^2 + 2cx\cos\phi\cos\gamma - c^2\cos^2\phi + c^2\sin^2\gamma. \qquad (2)$$

To pass from x, the variable employed by Mr. Hill, to Maxwell's θ, representing the angle AOQ in the plane of the base in Fig. 3,

$$x = CD - ON\sin\phi = c\cos\phi\cos\gamma - a\sin\phi\cos\theta, \qquad (3)$$

$$PQ^2 = r^2 + c^2 - 2rx = A^2 + 2Aa\cos\theta + a^2 + b^2, \qquad (4)$$

$$\tfrac{1}{2}(PA^2 + PB^2) = r^2 - 2cr\cos\phi\cos\gamma + c^2 = A^2 + a^2 + b^2, \qquad (5)$$

$$LN = MN\cos\phi + PM\sin\phi = (A + a\cos\theta)\cos\phi + b\sin\phi, \qquad (6)$$

$$PL = MN\sin\phi - PM\cos\phi = (A + a\cos\theta)\sin\phi - b\cos\phi, \qquad (7)$$

$$PN^2 = (A + a\cos\theta)^2 + b^2, \ldots\ldots \qquad (8)$$

Then

$$\frac{1}{2\,G\rho}\frac{dY}{d\theta} = J_1\,a\sin\phi\sin\theta - \frac{a\sin\theta}{PQ}\,a\sin\phi\sin\theta$$

$$= -a\sin\phi\frac{d}{d\theta}(J_1\cos\theta) + a\sin\phi\cos\theta\frac{dJ_1}{d\theta} - \sin\phi\frac{QN^2}{PQ}. \qquad (9)$$

Now

$$\frac{dJ_1}{d\theta} = \frac{d}{d\theta} \operatorname{sh}^{-1} \frac{QN}{PQ} = \frac{QN}{PQ} \left(\frac{\cos\theta}{\sin\theta} + \frac{A + a\cos\theta}{PN^2} a\sin\theta \right)$$

$$= \frac{a\cos\theta}{PQ} + \frac{QN^2}{PN^2} \cdot \frac{A + a\cos\theta}{PQ}, \qquad (10)$$

$$\frac{1}{2G\rho} \frac{dY}{d\theta} = - a\sin\phi \frac{d}{d\theta}(J_1 \cos\theta) + \sin\phi \frac{a^2 \cos^2\theta}{PQ}$$

$$+ \sin\phi \frac{QN^2}{PN^2} \frac{(A + a\cos\theta) a\cos\theta}{PQ} - \sin\phi \frac{QN^2}{PQ}, \qquad (11)$$

$$\frac{1}{2G\rho \sin\phi} \frac{dY}{d\theta} =$$

$$- a\frac{d}{d\theta}(J_1 \cos\theta) + \frac{a^2 \cos^2\theta}{PQ} + \frac{QN^2}{PN^2} \frac{(A + a\cos\theta) a\cos\theta}{PQ} - \frac{QN^2}{PQ}$$

$$= \quad \star \quad + \quad \star \quad + \frac{QN^2}{PN^2} \cdot \frac{PN^2 - b^2 - A(A + a\cos\theta)}{PQ} - \frac{QN^2}{PQ}$$

$$= \quad \star \quad + \quad \star \quad - \frac{QN^2}{PN^2} \cdot \frac{b^2}{PQ} - \left(A\frac{dJ_1}{d\theta} - \frac{A a\cos\theta}{PQ} \right)$$

$$= -\frac{d}{d\theta}(A + a\cos\theta) J_1 + \frac{a^2}{PQ} + \frac{A a\cos\theta}{PQ} - \frac{QN^2}{PQ} - \frac{QN^2}{PN^2} \cdot \frac{b^2}{PQ}, \qquad (12)$$

the star representing a term repeated.

Now J_1 vanishes at the limits, A and B, so that

$$\frac{Y}{G\rho \sin\phi} = \int_0^{2\pi} \frac{a^2 \, d\theta}{PQ} + \int \frac{A a\cos\theta \, d\theta}{PQ} - \int \frac{QN^2 \, d\theta}{PQ} - b \int \frac{QN^2}{PN^2} \cdot \frac{b \, d\theta}{PQ}, \qquad (13)$$

consisting of complete elliptic integrals, of the first and second kind, and the last term of the third kind, representing geometrically the apparent area Ω of the base AB as seen from P; as is proved by a dissection in Fig. 3 into line elements QNQ' of length $2a\sin\theta$ and breadth $a\sin\theta \, d\theta$; the apparent area $d\Omega$ at P of the element being

$$d\Omega = \frac{2a\sin\theta}{PQ} \cdot \frac{b}{PN} \cdot \frac{a\sin\theta \, d\theta}{PN} = 2\frac{QN^2}{PN^2} \cdot \frac{b}{PQ}. \qquad (14)$$

Thus

$$\frac{Y}{G\rho} = (Pa - QA - S - \Omega b) \sin\phi, \qquad (15)$$

$$P = \int_0^{2\pi} \frac{a \, d\theta}{PQ}, \quad Q = \int \frac{- a\cos\theta \, d\theta}{PQ}, \quad S = \int \frac{QN^2 \, d\theta}{PQ}, \left. \vphantom{\int} \right\} \quad (16)$$

$$\Omega = \int \frac{QN^2}{PN^2} \cdot \frac{b \, d\theta}{PQ},$$

and on reference to the *Trans. A. M. S.*, P is the potential of the circumference of the circular base AB, and $2\pi QA$ is Maxwell's M of § 701, *E. and M.*

3. The attraction at P of the line element QQ' of the circular base AB in Fig. 3 is along PN, and by (1), § 1, is equal to

$$\frac{QQ'}{PN.PQ}, \text{ having components } \frac{QQ'.PM}{PN^2.PQ} \text{ and } \frac{QQ'.MN}{PN^2.PQ}, \tag{1}$$

perpendicular and parallel to the base; so that, W denoting the potential at P of the base AB, covered with superficial density σ,

$$-\frac{1}{G\sigma}\frac{dW}{db} = \int_0^\pi \frac{QQ'.PM}{PN^2.PQ} a \sin\theta\, d\theta = 2\int \frac{QN^2}{PN^2}\cdot\frac{b\,d\theta}{PQ} = \Omega, \tag{2}$$

$$-\frac{1}{G\sigma}\frac{dW}{dA} = \int \frac{QQ'.MN}{PN^2.PQ} a \sin\theta\, d\theta = 2\int \frac{QN^2}{PN^2}\cdot\frac{A+a\cos\theta}{PQ}\, d\theta$$

$$= 2\int \left(\frac{dJ_1}{d\theta} - \frac{a\cos\theta}{PQ}\right) d\theta = Q, \tag{3}$$

since J_1 vanishes at both limits, A and B; also

$$\frac{1}{G\sigma}\frac{dW}{da} = P, \tag{4}$$

the potential of the circumference of the circle AB.

Then, since W is a homogeneous function of the first degree in a, A, b,

$$W = a\frac{dW}{da} + A\frac{dW}{dA} + b\frac{dW}{db}, \tag{5}$$

$$\frac{W}{G\sigma} = Pa - QA - \Omega b, \tag{6}$$

and we can write

$$\frac{Y}{G\rho} = \left(\frac{W}{G\sigma} - S\right)\sin\phi, \tag{7}$$

and it is shown in the sequel, (13), (14), § 13, that

$$S = -\tfrac{1}{3}\frac{PA^2.PB^2}{Ab}\frac{d\Omega}{dA} = \tfrac{2}{3}Pa - \tfrac{1}{3}Q\frac{A^2+a^2+b^2}{A}. \tag{8}$$

Interpreted in electromagnetism, Ω is the magnetic potential of the base AB magnetized normally, or of unit current circulating round the rim, while Q is the potential when the circle is magnetized parallel to AB; and $2\pi QA$ is the vector potential on Stokes current function of Ω (*E. and M.*, § 703).

Along the axis CO, $\phi = 0$, $A = 0$, $PQ = PA$:

$$\Omega = 2\pi \left(1 - \frac{PO}{PA} \right), \tag{9}$$

$$\frac{Y}{G\rho \sin \phi} = 2 \int_0^\pi \frac{a^2 d\theta}{PA} + 2 \int \frac{A a \cos \theta \, d\theta}{PA}$$

$$- 2 \int \frac{a^2 \sin^2 \theta \, d\theta}{PA} - 2\pi \, PO \left(1 - \frac{PO}{PA} \right)$$

$$= \frac{2\pi a^2}{PA} + 0 - \frac{\pi a^2}{PA} - 2\pi \cdot PO + 2\pi \frac{PO^2}{PA} = \pi \frac{(PA - PO)^2}{PA}. \tag{10}$$

Thus a smooth particle on the surface will make $n/2\pi \sim$ /sec., about the position of equilibrium, at the vertex K or at O, where

$$n^2 = G\rho \pi \frac{(KA - KO)^2}{KA \cdot KC}, \quad \text{or} \quad G\rho \pi \frac{OA}{OC}. \tag{11}$$

4.   The component attraction of the segment QRQ' in the direction PC is

$$\int \frac{PL}{Pn} \cdot \frac{2nq \, dy}{PQ \cdot Pn} = 2 \frac{PL}{PQ} \int \frac{\sqrt{(LQ^2 - y^2)} \, dy}{PL^2 + y^2}$$

$$= 2 \int \frac{PL \cdot PQ \cdot dy}{(PL^2 + y^2)\sqrt{(LQ^2 - y^2)}} - 2 \frac{PL}{PQ} \int \frac{dy}{\sqrt{(LQ^2 - y^2)}}$$

$$= 2 J_3 - 2 \frac{PL}{PQ} J_2, \tag{1}$$

where

$$J_2 = \cos^{-1} \frac{LN}{LQ} = \sin^{-1} \frac{QN}{LQ} = \tan^{-1} \frac{QN}{LN}, \tag{2}$$

$$J_3 = \cos^{-1} \frac{PQ \cdot LN}{PN \cdot LQ} = \sin^{-1} \frac{PL \cdot NQ}{PN \cdot LQ} = \tan^{-1} \frac{PL \cdot NQ}{PQ \cdot LN}; \tag{3}$$

and J_2 is QLN, the angle between the planes PLQ, PLA; while J_3 is the complement of the angle between the planes PQN, PQL.

Then, in Mr. Hill's notation,

$$\frac{1}{2 G\rho} \frac{dX'}{dx} = J_3 - \frac{PL}{PQ} J_2, \tag{4}$$

and X' will represent X, the component attraction along PC, provided PC does not cut the base AB; otherwise a modification is required which is given in the sequel. We divide X' into two parts, X_1 and X_2, where

$$\frac{1}{2 G\rho} \frac{dX_1}{dx} = J_3, \qquad \frac{1}{2 G\rho} \frac{dX_2}{dx} = - \frac{PL}{PQ} J_2; \tag{5}$$

and then

$$\frac{1}{2\,G\rho\sin\phi}\,\frac{dX_1}{d\theta} = a\,J_3\sin\theta = -\frac{d}{d\theta}\,(a\,J_3\cos\theta) + a\cos\theta\,\frac{dJ_3}{d\theta}, \qquad (6)$$

and, with $LQ^2 = c^2 - x^2$,

$$\frac{dJ_3}{d\theta} = \frac{d}{d\theta}\sin^{-1}\frac{PL\,.\,NQ}{PN\,.\,LQ}$$

$$= \frac{PL\,.\,NQ}{PQ\,.\,LN}\left[\frac{-a\sin\theta\sin\phi}{PL} + \frac{(A + a\cos\theta)\,a\sin\theta}{PN^2} + \frac{\cos\theta}{\sin\theta} + \frac{x\dfrac{dx}{d\theta}}{LQ^2}\right], \qquad (7)$$

in which, with

$$PL = (A + a\cos\theta)\sin\phi - b\cos\phi, \quad LN = (A + a\cos\theta)\cos\phi + b\sin\phi, \qquad (8)$$

$$\frac{-a\sin\phi\sin\theta}{PL} + \frac{(A + a\cos\theta)\,a\sin\theta}{PN^2} = -b\,\frac{LN\,.\,a\sin\theta}{PL\,.\,PN^2}, \qquad (9)$$

$$\frac{\cos\theta}{\sin\theta} + \frac{x\dfrac{dx}{d\theta}}{LQ^2} = \frac{LN\,[a\cos\phi + (A\cos\phi + b\sin\phi)\cos\theta]}{\sin\theta\,LQ^2}$$

$$= \frac{LN\,(c\cos\phi - x\cos\gamma)}{\sin\gamma\sin\theta\,LQ^2}, \qquad (10)$$

so that

$$\frac{dJ_3}{d\theta} = -\frac{a^2\sin^2\theta}{LN^2}\cdot\frac{b}{PQ} + \frac{a}{\sin\gamma}\,\frac{PL\,(c\cos\phi - x\cos\gamma)}{LQ^2\,.\,PQ}$$

$$= -\tfrac{1}{2}\frac{d\Omega}{d\theta} + c\,\frac{(r - x)\,(c\cos\phi - x\cos\gamma)}{(c^2 - x^2)\,PQ} \qquad (11)$$

and, with

$$a\sin\phi\cos\theta = c\cos\phi\cos\gamma - x, \qquad (12)$$

$$\frac{1}{2\,G\rho}\,\frac{dX_1}{d\theta} =$$

$$-\sin\phi\,\frac{d}{d\theta}\,(a\,J_3\cos\theta) + a\sin\phi\cos\theta\left[-\frac{QN^2}{PN^2}\cdot\frac{b}{PQ} + c\,\frac{(r-x)(c\cos\phi - x\cos\gamma)}{LQ^2\,.\,PQ}\right]$$

$$= \qquad \star \qquad + \sin\phi\left(-b\,\frac{dJ_1}{d\theta} + \frac{ab\cos\theta}{PQ} + \tfrac{1}{2}A\,\frac{d\Omega}{d\theta}\right)$$

$$+ c\,\frac{(c\cos\phi\cos\gamma - x)\,(r - x)}{c^2 - x^2}\cdot\frac{c\cos\phi - x\cos\gamma}{PQ}$$

$$= \star + \star - c\,\frac{c\cos\phi - x\cos\gamma}{PQ} + \frac{\tfrac{1}{2}c(1 + \cos\phi\cos\gamma)(r + c)}{c + x}\cdot\frac{c\cos\phi - x\cos\gamma}{PQ}$$

$$- \frac{\tfrac{1}{2}c(1 - \cos\phi\cos\gamma)(r - c)}{c - x}\cdot\frac{c\cos\phi - x\cos\gamma}{PQ}, \qquad (13)$$

so that three elliptic integrals of the third kind appear in this expression for X_1.

But if P' is the point on CP inverse to P with respect to the spherical surface

$$CP' = \frac{c^2}{r}, \quad P'L = \frac{c^2}{r} - x, \quad \text{and} \quad \frac{P'Q}{PQ} = \frac{c}{r}, \tag{14}$$

and with

$$J_3' = \cos^{-1} \frac{P'Q \cdot LN}{P'N \cdot LQ} = \sin^{-1} \frac{P'L \cdot NQ}{P'N \cdot LQ}, \tag{15}$$

$$
\begin{aligned}
\frac{dJ_3'}{d\theta} &= -\tfrac{1}{2} \frac{d\Omega'}{d\theta} + c \frac{\left(\frac{c^2}{r} - x\right)(c \cos\phi - x \cos\gamma)}{LQ^2 \cdot P'Q} \\
&= -\tfrac{1}{2} \frac{d\Omega'}{d\theta} + \frac{(c^2 - xr)(c \cos\phi - x \cos\gamma)}{(c^2 - x^2) PQ},
\end{aligned} \tag{16}
$$

where Ω' denotes the apparent area of the base AB at P'; and then

$$\frac{d}{d\theta}(J_3 + \tfrac{1}{2}\Omega + J_3' + \tfrac{1}{2}\Omega') = \frac{r+c}{c+x} \cdot \frac{c \cos\phi - x \cos\gamma}{PQ}, \tag{17}$$

$$\frac{d}{d\theta}(J_3 + \tfrac{1}{2}\Omega - J_3' - \tfrac{1}{2}\Omega') = \frac{r-c}{c-x} \cdot \frac{c \cos\phi - x \cos\gamma}{PQ}, \tag{18}$$

$$
\begin{aligned}
\frac{1}{2G\rho} \frac{dX_1}{d\theta} =\ & -\sin\phi \frac{d}{d\theta}(aJ_3 \cos\theta) - b \sin\phi \frac{dJ_1}{d\theta} + \sin\phi \frac{ab \cos\theta}{PQ} - c \frac{c \cos\phi - x \cos\gamma}{PQ} \\
& + \tfrac{1}{2}(b \cos\phi + r) \frac{d\Omega}{d\theta} + \tfrac{1}{2} c \frac{d\Omega'}{d\theta} + c \cos\phi \cos\gamma \frac{dJ_3}{d\theta} + c \frac{dJ_3'}{d\theta}.
\end{aligned} \tag{19}
$$

For the calculation of X_2,

$$\frac{1}{2G\rho} \frac{dX_2}{dx} = -\frac{PL}{PQ} J_2, \tag{20}$$

and

$$\int \frac{PL}{PQ} dx = \int \frac{r-x}{(r^2 + c^2 - 2rx)^{\frac{3}{2}}} dx = \frac{rx - 2r^2 + c^2}{3r^2} PQ, \tag{21}$$

so that

$$\frac{1}{2G\rho} \frac{dX_2}{d\theta} = -\frac{d}{d\theta}\left(\frac{rx - 2r^2 + c^2}{3r^2} \cdot PQ \cdot J_2\right) + \frac{rx - 2r^2 + c^2}{3r^2} PQ \frac{dJ_2}{d\theta}, \tag{22}$$

where

$$J_2 = \tan^{-1}\frac{QN}{LN} = \tan^{-1}\frac{a\sin\theta}{(a\cos\theta + A)\cos\phi + b\sin\phi},$$
$$\frac{dJ_2}{d\theta} = a\frac{(A\cos\theta + c)\cos\phi + b\cos\theta\sin\phi}{LQ^2} = c\frac{c\cos\phi - x\cos\gamma}{LQ^2}. \quad \left.\right\} \quad (23)$$

$$\frac{1}{2G\rho}\frac{dX_2}{d\theta} = -\frac{d}{d\theta}\left(\frac{rx - 2r^2 + c^2}{3r^2} . PQ . J_2\right)$$

$$+ \frac{c}{3r^2} . \frac{(rx - 2r^2 + c^2)(-2rx + r^2 + c^2)}{c^2 - x^2} . \frac{c\cos\phi - x\cos\gamma}{PQ}$$

$$= \quad \bigstar \quad + \frac{c}{3r^2}\left[2r^2 - \frac{(2r-c)(r+c)^3}{2c(c+x)} - \frac{(2r+c)(r-c)^3}{2c(c-x)}\right]\frac{c\cos\phi - x\cos\gamma}{PQ}$$

$$= \quad \bigstar \quad + \frac{2}{3}c\frac{c\cos\phi - x\cos\gamma}{PQ} - \frac{(2r-c)(r+c)^2}{6r^2}\frac{d}{d\theta}(J_3 + \tfrac{1}{2}\Omega + J_3' + \tfrac{1}{2}\Omega')$$

$$- \frac{(2r+c)(r-c)^2}{6r^2}\frac{d}{d\theta}(J_3 + \tfrac{1}{2}\Omega - J_3' - \tfrac{1}{2}\Omega')$$

$$= \quad \bigstar \quad + \quad \bigstar \quad - \frac{2}{3}r\frac{d}{d\theta}(J_3 + \tfrac{1}{2}\Omega) - \left(c - \frac{c^3}{3r^2}\right)\frac{d}{d\theta}(J_3' + \tfrac{1}{2}\Omega'), \quad (24)$$

the star replacing a term repeated.

Then, by addition,

$$\frac{1}{2G\rho}\frac{dX'}{d\theta} = -b\sin\phi\frac{dJ_1}{d\theta} - \frac{d}{d\theta}\left(\frac{rx - 2r^2 + c^2}{3r^2} . PQ . J_2\right) - \frac{d}{d\theta}(r - x . J_3)$$

$$- \tfrac{1}{3}a\cos\phi\frac{a}{PQ} - \tfrac{1}{3}(A\cos\phi - 2b\sin\phi)\frac{a\cos\theta}{PQ}$$

$$+ \tfrac{1}{2}b\frac{d\Omega}{d\theta} + \tfrac{1}{3}r\frac{d}{d\theta}(J_3 + \tfrac{1}{2}\Omega) + \frac{c^3}{3r^2}\frac{d}{d\theta}(J_3 + \tfrac{1}{2}\Omega'). \quad (25)$$

So long as $\phi > \gamma$, $X' = X$, and J_2, J_3 vanish at both limits, as well as J_1; and

$$\frac{X}{G\rho} = -\tfrac{2}{3}a\cos\phi\int_0^\pi\frac{a\,d\theta}{PQ} + \tfrac{2}{3}(A\cos\phi - 2b\sin\phi)\int\frac{-a\cos\theta\,d\theta}{PQ}$$

$$+ \Omega b\cos\phi + \tfrac{1}{3}\Omega r + \tfrac{1}{3}\Omega'\frac{c^3}{r^2}.$$

$$= -\frac{W}{G\sigma}\cos\phi + \tfrac{2}{3}Pa\cos\phi - \tfrac{2}{3}Q(A\cos\phi + b\sin\phi)$$

$$+ \tfrac{1}{3}\Omega r + \tfrac{1}{3}\Omega'\frac{c^3}{r^2}. \quad (26)$$

5.   But when $\phi < \gamma$, CP cuts the base AB, and the attraction X'' must be added of the spherical segment cut off by the plane BB' perpendicular to CP; and X'' is obtained by replacing J_2 and J_3 by π in (4), § 4, and integrating from x_2 to c; then

$$\frac{1}{G\rho}\frac{dX''}{dx} = 2\pi\left(1 - \frac{PL}{PQ}\right), \tag{1}$$

$$\begin{aligned}\frac{X''}{G\rho} &= 2\pi\int_{x_2}^{c}\left[1 - \frac{r-x}{(r^2+c^2-2rx)^{\frac{1}{2}}}\right]dx \\ &= 2\pi(c-x_2) + 2\pi\frac{(2r+c)(r-c)^2}{3r^2} + 2\pi\frac{rx_2-2r^2+c^2}{3r^2}r_2.\end{aligned} \tag{2}$$

At the upper limit we must now take $J_3 = \pi$, $J_3' = -\pi$, and then

$$\frac{X}{G\rho} = \frac{X'+X''}{G\rho} = \text{same value as in (26), § 4,} \tag{3}$$

so that there is no discontinuity in X as P crosses the line CB.

As a verification for X, consider the case where P is at K or O, when $\phi = 0$:

$$\frac{X}{G\rho} = -\frac{2\pi a^3}{PA} + \Omega b + \tfrac{1}{3}\Omega r + \tfrac{1}{3}\Omega'\frac{c^3}{r^2}. \tag{4}$$

At K, we must take

$$\left.\begin{aligned}r = c, \quad b = -c(1-\cos\gamma), \quad \Omega = -2\pi(1-\sin\tfrac{1}{2}\gamma), \\ \Omega' = 4\pi + \Omega, \quad \Omega + \Omega' = 4\pi\sin\tfrac{1}{2}\gamma,\end{aligned}\right\} \tag{5}$$

$$\frac{X}{G\rho} = 4\pi c\sin^2\tfrac{1}{2}\gamma - \tfrac{8}{3}\pi c\sin^3\tfrac{1}{2}\gamma = 2\pi . KO\left(1 - \tfrac{2}{3}\frac{KO}{KB}\right), \tag{6}$$

agreeing with the result of a direct integration, as for X'' in (2).

At O, we take

$$r = x_2 = c\cos\gamma, \quad b = 0, \quad \Omega = 2\pi, \quad \Omega' = -2\pi(1-\sin\gamma), \tag{7}$$

$$\frac{X}{G\rho} = -\tfrac{2}{3}\pi a + \tfrac{2}{3}\pi r - \tfrac{2}{3}\frac{\pi c^2}{r^2}(c-a) = \tfrac{2}{3}\pi c\left(-\sin\gamma + \cos\gamma - \frac{1}{1+\sin\gamma}\right), \tag{8}$$

agreeing, when the sign is changed, with the value obtained by integration.

For a hemisphere, $\gamma = \tfrac{1}{2}\pi$,

$$\frac{X}{G\rho} = 2\pi c(1 - \tfrac{1}{3}\sqrt{2}), \text{ at } K, \quad = -\pi c, \text{ at } O. \tag{9}$$

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In the plane of the base of the hemisphere, $\phi = \tfrac{1}{2}\pi$,

$$\frac{X}{G\rho} = \tfrac{1}{3}\Omega r + \tfrac{1}{3}\Omega'\frac{c^3}{r^2}, \tag{10}$$

$$r > c, \quad \Omega = 0, \quad \Omega' = 2\pi, \quad \frac{X}{G\rho} = \frac{2\pi c^3}{3r^2}, \tag{11}$$

$$r < c, \quad \Omega = 2\pi, \quad \Omega' = 0, \quad \frac{X}{G\rho} = \tfrac{2}{3}\pi r, \tag{12}$$

a verification.

6. If V denotes the gravitation potential of the segment or flat lens,

$$\frac{1}{G\rho}\frac{dV}{dr} = -\frac{X}{G\rho} = \frac{W}{G\sigma}\cos\phi - \tfrac{2}{3}Pa\cos\phi$$
$$+ \tfrac{2}{3}Q(A\cos\phi + b\sin\phi) - \tfrac{1}{3}\Omega r - \tfrac{1}{3}\Omega'\frac{c^3}{r^2}, \tag{1}$$

$$\frac{1}{G\rho}\frac{dV}{r\,d\phi} = -\frac{Y}{G\rho} = -\frac{W}{G\sigma}\sin\phi + \tfrac{2}{3}Pa\sin\phi - \tfrac{1}{3}Q\frac{A^2 + a^2 + b^2}{A}\sin\phi, \tag{2}$$

$$\frac{1}{G\rho}\frac{dV}{dA} = -\frac{X}{G\rho}\sin\phi - \frac{Y}{G\rho}\cos\phi$$
$$= -\tfrac{1}{3}\Omega r\sin\phi - \tfrac{1}{3}\Omega'\frac{c^3}{r^2}\sin\phi + \tfrac{1}{3}Q(b + b')$$
$$= -\tfrac{1}{3}(\Omega A - Q b) - \tfrac{1}{3}\frac{c}{r}(\Omega' A' - Q' b') \tag{3}$$

(an accent referring to the point P'),

$$\frac{1}{G\rho}\frac{dV}{db} = \frac{X}{G\rho}\cos\phi - \frac{Y}{G\rho}\sin\phi$$
$$= \frac{W}{G\sigma} - \tfrac{1}{3}\Omega r\cos\phi - \tfrac{1}{3}\Omega'\frac{c^3}{r^2}\cos\phi + \tfrac{1}{3}Q(A + A') - \tfrac{2}{3}Pa$$
$$= \frac{W}{G\sigma} - \tfrac{1}{3}(\Omega r\cos\phi - QA) - \tfrac{1}{3}\frac{c}{r}(\Omega' r'\cos\phi - Q'A') - \tfrac{2}{3}Pa. \tag{4}$$

We shall find in the sequel, § 8, that these components are derivable from a potential V, where

$$\frac{V}{G\rho} = -\tfrac{1}{2}\frac{W}{G\sigma}b + \Omega(\tfrac{1}{2}c^2 - \tfrac{1}{6}r^2) + \tfrac{1}{3}\Omega'\frac{c^3}{r}$$
$$- \tfrac{1}{3}Pa A\cot\phi + \tfrac{1}{3}QAc\cos\gamma. \tag{5}$$

The cyclic constant 4π in Ω and Ω' will give rise to the terms

$$4\pi(\tfrac{1}{2}c^2 - \tfrac{1}{6}r^2), \text{ and } \tfrac{4}{3}\pi\frac{c^3}{r}, \tag{6}$$

in $\dfrac{V}{G\rho}$, corresponding to the potential, inside and outside, of a solid sphere.

The Attraction of a Spherical Bowl.

7. The bowl is supposed constituted of a superficial density σ over the spherical surface of the segment; and a similar dissection by planes perpendicular to CP will show that X_1, Y_1, the components of attraction, will be given by

$$\frac{1}{G\sigma}\frac{dY_1}{dx} = \text{component attraction perpendicular to } CP \text{ of the arc } QRQ'$$

in Fig. 2, allowing for the slanting section,

$$= \frac{2c \cdot QN}{PQ^3}, \tag{1}$$

$$\frac{1}{G\sigma}\frac{dY_1}{d\theta} = 2ac\sin\phi\sin\theta\,\frac{a\sin\theta}{PQ^3}$$

$$= \frac{2ac}{A}\sin\phi\frac{d}{d\theta}\left(\frac{a\sin\theta}{PQ}\right) - 2\frac{c}{A}\sin\phi\frac{a\cos\theta}{PQ}, \tag{2}$$

$$\frac{Y_1}{G\sigma} = 2\frac{c}{A}\sin\phi\int_0^\pi\frac{-a\cos\theta\,d\theta}{PQ} = Q\frac{c}{r}, \tag{3}$$

$$\frac{1}{G\sigma}\frac{dX_1'}{dx} = \frac{2c \cdot PL \cdot J_2}{PQ^3} = \frac{2c(r-x)J_2}{(r^2+c^2-2rx)^{\frac{3}{2}}}$$

$$= -\frac{2c}{r^2}\frac{d}{dx}\left(\frac{c^2-rx}{PQ} \cdot J_2\right) + \frac{2c}{r^2} \cdot \frac{c^2-rx}{PQ}\frac{dJ_2}{dx}, \tag{4}$$

$$\frac{1}{G\sigma}\frac{dX_1'}{d\theta} = -\frac{2c}{r^2}\frac{d}{d\theta}\left(\frac{c^2-rx}{PQ} \cdot J_2\right) + \frac{c^2}{r^2}\frac{d}{d\theta}(\Omega' + 2J_3'). \tag{5}$$

Then, if $\phi > \gamma$, $X_1' = X_1$, and J_2, J_3' are both zero at B, so that

$$\frac{X_1}{G\sigma} = \Omega'\frac{c^2}{r^2}. \tag{6}$$

But if $\phi < \gamma$, X_1'' must be added to X_1', where X'' is obtained by putting $J_2 = \pi$ in (4),

$$\frac{1}{G\sigma}\frac{dX_1''}{dx} = \frac{2\pi c \cdot PL}{PQ^3}, \tag{7}$$

$$\frac{X_1''}{G\sigma} = 2\pi c\int_{x_2}^c\frac{r-x}{(r^2+c^2-2rx)^{\frac{3}{2}}}\,dx = \frac{2\pi c}{r^2}\left(\frac{c^2-rx_2}{PB}+c\right), \tag{8}$$

and with $J_3 = -\pi$ at B,

$$\frac{X_1}{G\sigma} = \frac{X_1' + X_1''}{G\sigma} = \Omega'\frac{c^2}{r^2}, \tag{9}$$

as before in (6), so that there is no discontinuity in X_1 as P crosses CB.

At the vertex K, where $\phi = 0$, $A = 0$,

$$PQ^2 = A^2 + a^2 + b^2 + 2Aa\cos\theta = KA^2\left(1 + \frac{2Aa}{KA^2}\cos\theta\right), \tag{10}$$

$$\frac{Y_1}{G\sigma} = \frac{2ac\sin\phi}{A.KA}\int_0^\pi\left(-\cos\theta + \frac{Aa\cos^2\theta}{KA^2}\dots\right)d\theta = \frac{\pi a^2 c}{KA^3}\sin\phi, \tag{11}$$

so that the small oscillation of a smooth particle on the surface of the bowl near K is given by

$$\frac{d^2\phi}{dt^2} + n^2\sin\phi = 0, \text{ where } n^2 = \frac{Y}{c\sin\phi} = G\sigma\frac{\pi a^2}{KA^3}, \tag{12}$$

and the number of oscillations for second is $n \div 2\pi$.

The Potential of the Bowl and the Spherical Segment, a Flat Lens.

8. With the same dissection, denoting the potential of the bowl by U,

$$\frac{1}{G\sigma}\frac{dU'}{dx} = \text{potential at } P \text{ of the arc } QRQ' \times \sec CRL$$

$$= \frac{2c.J_2}{PQ} = -\frac{2c}{r}\frac{d}{dx}(PQ.J_2) + \frac{2c}{r}PQ\frac{dJ_2}{dx}, \tag{1}$$

$$\frac{1}{G\sigma}\frac{dU'}{d\theta} = -\frac{2c}{r}\frac{d}{d\theta}(PQ.J_2) + \frac{2c^2}{r}\cdot\frac{c\cos\phi - x\cos\gamma}{c^2 - x^2}\cdot PQ$$

$$= \bigstar + \frac{2c^2}{r}\left[\frac{(r+c)^2}{2c(c+x)} + \frac{(r-c)^2}{2c(c-x)}\right]\frac{c\cos\phi - x\cos\gamma}{PQ}$$

$$= \bigstar + c\frac{r+c}{r}\frac{d}{d\theta}(J_3 + \tfrac{1}{2}\Omega + J_3' + \tfrac{1}{2}\Omega')$$

$$\qquad + c\frac{r-c}{r}\frac{d}{d\theta}(J_3 + \tfrac{1}{2}\Omega - J_3' - \tfrac{1}{2}\Omega')$$

$$= \bigstar + c\frac{d}{d\theta}(2J_3 + \Omega) + \frac{c^2}{r}\frac{d}{d\theta}(2J_3' + \Omega'). \tag{2}$$

With $\phi > \gamma$, $U = U'$, and J_2, J_3, J_3' are zero at B;

$$\frac{U}{G\sigma} = \Omega c + \Omega'\frac{c^2}{r}. \tag{3}$$

But with $\phi < \gamma$, U'' must be added to U' to obtain U, where, replacing J_2 by π in (1),

$$\frac{1}{G\sigma} \frac{d U''}{dr} = \frac{2\pi c}{PQ}, \qquad \frac{U''}{G\sigma} = 2\pi c \int_{x_2}^{c} \frac{dx}{PQ} = \frac{2\pi c}{r}(r_2 - r + c); \qquad (4)$$

and now J_3 becomes π at B, and J_3' becomes $-\pi$; and then

$$\frac{U}{G\sigma} = \Omega c + \Omega' \frac{c^3}{r}, \qquad (5)$$

so that there is no discontinuity in U as P crosses CB.

Expressed by its mass $M = 2\pi\sigma c . OK$,

$$\frac{U}{GM} = \frac{1}{2\pi . OK}\left(\Omega + \Omega' \frac{c}{r}\right). \qquad (6)$$

We can now determine the potential V of the segment in a simple manner by considering it as a homogeneous function in the second degree of the dimensions, and by calculating the variation when the figure swells by variation of r and c, γ and ϕ remaining constant; and then

$$2V = r\frac{dV}{dr} + c\frac{dV}{dc} = -Xr + c\frac{dV}{dc}. \qquad (7)$$

Now the variation dV, due to the change dc in c, is equivalent to the addition of the potential of a spherical bowl on the surface, of thickness dc and superficial density $\rho\, dc$, less the potential of a circular disc of radius $a = c \sin \gamma$, thickness $dc \cos\gamma$, and superficial density $\rho\, dc \cos\gamma$; so that, from (6), §3, and (3), §8,

$$\frac{1}{G\rho} \frac{dV}{dc} = \Omega c + \Omega' \frac{c^2}{r} - \frac{W}{G\sigma} \cos\gamma, \qquad (8)$$

$$\frac{2V}{G\rho} = \frac{W}{G\sigma} r \cos\phi - \tfrac{2}{3} Par \cos\phi + \tfrac{2}{3} Qr (A \cos\phi + b \sin\phi)$$

$$- \tfrac{1}{3}\Omega r^2 - \tfrac{1}{3}\Omega' \frac{c^3}{r} + \Omega c^2 + \Omega' \frac{c^3}{r} - \frac{W}{G\sigma} c \cos\gamma, \qquad (9)$$

which is equivalent to the statement in (5), §6.

9. The potential V may be calculated directly, by the method employed for X and Y, from the relation

$$\frac{1}{G\rho} \frac{dV}{dx} = \text{potential of the circular segment } QRQ'$$
$$\text{at a point } P \text{ on its axis } CP. \qquad (1)$$

The potential at P of the line element qq' is

$$\int_{-nq}^{nq} \frac{dz}{Pn} = \int \frac{dz}{\sqrt{(Pn^2 + z^2)}} = \log \frac{Pq + nq}{Pq - nq} = 2 \, \text{th}^{-1} \frac{nq}{PQ}, \qquad (2)$$

since $Pq = PQ$ round the periphery of the circle in Fig. 2.

The potential then at P of the segment QRQ' is

$$2 \int_{LN}^{LR} \text{th}^{-1} \frac{\sqrt{(LQ^2 - y^2)}}{PQ} \, dy = 2y \, \text{th}^{-1} \frac{nq}{PQ} \Big]_{LN}^{LR} - 2 \int y \frac{- y \, dy \cdot PQ}{(PQ^2 - LQ^2 + y^2) \sqrt{(LQ^2 - y^2)}}$$

$$= -2LN \text{th}^{-1} \frac{NQ}{PQ} + 2 PQ \int \frac{dy}{\sqrt{(LQ^2 - y^2)}} - 2 \int \frac{PQ \cdot PL^2 \, dy}{(PL^2 + y^2) \sqrt{(LQ^2 - y^2)}}$$

$$= -2LN \text{ch}^{-1} \frac{PQ}{PN} + 2 PQ \cos^{-1} \frac{LN}{LQ} - 2 PL \cos^{-1} \frac{PQ \cdot LN}{PN \cdot LQ}, \qquad (3)$$

or, in the previous notation,

$$\frac{1}{G\rho} \frac{dV}{dx} = -2 LN \cdot J_1 + 2 PQ \cdot J_2 - 2 PL \cdot J_3. \qquad (4)$$

Thence V was found by an integration by parts similar to those above; this was the method employed at first, but the work was very heavy, and it is omitted here, as the result obtained by the short method in (7), § 8, was found to be in agreement.

10. The well-known expression for the potential V of the homogeneous ellipsoid

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1 \qquad (1)$$

may be interpreted in the same way, by treating V as a homogeneous function of the second degree in the linear scale of the ellipsoid, and in x, y, z, the coordinates of the point P; so that

$$2 V = a \frac{dV}{da} + x \frac{dV}{dx} + y \frac{dV}{dy} + z \frac{dV}{dz}$$

$$= a \frac{dV}{da} - x X - y Y - z Z. \qquad (2)$$

Taking the components of attraction, X, Y, Z, as known,

$$\frac{X, Y, Z}{G\rho} = 2 x A_\lambda, \quad 2 y B_\lambda, \quad 2 z C_\lambda, \qquad (3)$$

$$A_\lambda, B_\lambda, C_\lambda = \int_\lambda^\infty \frac{2 \pi a b c}{(\lambda + a^2, \lambda + b^2, \lambda + c^2)} \cdot \frac{d\lambda}{\sqrt{P}}, \qquad (4)$$

$$P = 4 \cdot \lambda + a^2 \cdot \lambda + b^2 \cdot \lambda + c^2, \quad A_\lambda + B_\lambda + C_\lambda = \frac{4 \pi a b c}{\sqrt{P}}; \qquad (5)$$

the variation dV due to the change da may be considered the potential of a film in electrical equilibrium, of superficial density $\rho \dfrac{p}{a} da$, where p is the perpendicular from the centre on the tangent plane; and the theorem

$$a \frac{dV}{da} = \int_\lambda^\infty \frac{4\pi\, G\rho\, abc\, d\lambda}{\sqrt{P}}, \qquad (6)$$

being taken as known, then

$$\frac{V}{G\rho} = \int_\lambda^\infty \left(1 - \frac{x^2}{\lambda + a^2} - \frac{y^2}{\lambda + b^2} - \frac{z^2}{\lambda + c^2}\right) \frac{2\pi\, abc\, d\lambda}{\sqrt{P}} \qquad (7)$$

is the result in consequence.

The Stokes Current Function.

11. In these investigations of the attraction of a body symmetrical about an axis Ox, the current function is useful, invented by Stokes (*Cambridge Phil. Soc. Trans.*, 1842). Denoting it by N, for a potential V, it satisfies the relations

$$\frac{dN}{dx} = 2\pi y \frac{dV}{dy}, \quad \frac{dN}{dy} = -2\pi y \frac{dV}{dx}, \qquad (1)$$

$$\frac{d}{dx}\left(\frac{1}{y}\frac{dN}{dx}\right) + \frac{d}{dy}\left(\frac{1}{y}\frac{dN}{dy}\right) = 0, \qquad (2)$$

while

$$\frac{d}{dx}\left(y\frac{dV}{dx}\right) + \frac{d}{dy}\left(y\frac{dV}{dy}\right) = 0, \text{ or } -4\pi\, G\rho. \qquad (3)$$

Thus in Maxwell's *E. and M.*, § 703, $M = 2\pi Q A$ is the current function of the magnetic potential Ω.

If the ellipsoid in (1), § 10, is of revolution about Ox, $b = c$,

$$\frac{V}{G\rho} = \int_\lambda^\infty \left(1 - \frac{x^2}{\lambda + a^2} - \frac{y^2 + z^2}{\lambda + b^2}\right) \frac{\pi\, ab^2\, d\lambda}{(\lambda + b^2)\sqrt{(\lambda + a^2)}}, \qquad (4)$$

and we find

$$\frac{N}{2\pi\, G\rho} = -xy^2 B_\lambda - \tfrac{2}{3} \frac{\pi\, ab^2\, x^3}{(\lambda + a^2)^{\frac{3}{2}}}$$

$$= \tfrac{1}{2} xy^2 A_\lambda - \frac{\pi\, ab^2\, xy^2}{(\lambda + b^2)\sqrt{(\lambda + a^2)}} - \tfrac{2}{3} \frac{\pi\, ab^2\, x^3}{(\lambda + a^2)^{\frac{3}{2}}}, \qquad (5)$$

$$\frac{1}{2\pi\, G\rho} \frac{dN}{dx} = -2y^2 B_\lambda = y\frac{dV}{dy}, \quad \frac{1}{2\pi\, G\rho} \frac{dN}{dy} = 2xy A_\lambda = -y\frac{dV}{dx}. \qquad (6)$$

Thus the velocity function for motion of the ellipsoid along Ox with velocity U being

$$\phi = -\frac{Ux\,A_\lambda}{B_0 + C_0}, \quad \text{or} \quad -\frac{Ux\,A_\lambda}{2\,B_0}, \quad \text{when } B_0 = C_0, \tag{7}$$

the current function is

$$\psi = -\frac{U y^2\, B_\lambda}{2\,B_0}, \tag{8}$$

and when the ellipsoid is reduced to rest, and the fluid streams part,

$$\phi = -Ux\left(1 + \frac{A_\lambda}{2\,B_0}\right), \quad \psi = \tfrac{1}{2}U y^2\left(1 - \frac{B_\lambda}{B_0}\right), \tag{9}$$

making $\psi = 0$ over the surface.

In the magnetic analogue, the spheroid, of soft iron of magnetic permeability μ, held with its axis parallel to a uniform magnetic field of strength X, will have

$$\phi = Xx\left[-1\frac{(\mu-1)A_\lambda}{\mu A_0 + 2\,B_0}\right], \quad \psi = \tfrac{1}{2}X y^2\left[1 + \frac{2(\mu-1)B_\lambda}{\mu A_0 + 2\,B_0}\right] \tag{10}$$

in the exterior field, and in the interior

$$\phi = \frac{4\pi Xx}{\mu A_0 + 2\,B_0}, \quad \psi = \frac{2\pi\mu X y^2}{\mu A_0 + 2\,B_0}; \tag{11}$$

reducing for $\mu = 0$ to the hydrodynamical case of (9).

For an oblate spheroid, putting

$$\lambda + a^2 = (b^2 - a^2)\cot^2\theta, \quad \lambda + b^2 = (b^2 - a^2)\csc^2\theta, \tag{12}$$

$$A_\lambda = \frac{2\pi a b^2}{(b^2 - a^2)^{\frac{3}{2}}}(\tan\theta - \theta), \quad A_0 = \frac{2\pi b^2}{(b^2 - a^2)^{\frac{3}{2}}}\left[\sqrt{(b^2 - a^2)} - a\cos^{-1}\frac{a}{b}\right], \tag{13}$$

$$B_\lambda = \frac{\tfrac{1}{2}\pi a b^2}{(b^2 - a^2)^{\frac{3}{2}}}(2\theta - \sin 2\theta), \quad B_0 = \frac{\pi a}{(b^2 - a^2)^{\frac{3}{2}}}\left[b^2\cos^{-1}\frac{a}{b} - a\sqrt{(b^2 - a^2)}\right]. \tag{14}$$

If v denotes the potential when the spheroid is insulated and electrified by a charge Q to potential v_0, with $G\sigma_0$ the electric density at A,

$$\frac{v}{G\sigma_0} = \frac{1}{G\rho}\frac{dV}{da} = 4\pi b\theta, \quad \frac{Q}{G\sigma_0} = \int\frac{p}{a}dS = 4\pi b^2, \tag{15}$$

so that the capacity is $b/\cos^{-1}\frac{a}{b}$.

For a disc, with $a = 0$,

$$A_0 = 2\pi, \quad B_0 = 0, \quad \frac{B_0}{a} = \frac{\pi^2}{2b}, \quad \frac{a A_0}{2 B_0} = \frac{2b}{\pi}, \tag{16}$$

$$\sin\theta = \frac{AB}{PA + PB}, \tag{17}$$

and the capacity is $b/\tfrac{1}{2}\pi$ (*E. and M.*, § 151).

12. If L denotes the current function corresponding to the potential W of the circular disc AB, it was found in *Trans. A. M. S.*, pp. 501, 513,

$$\frac{L}{2\pi G\sigma} = -\tfrac{1}{2}Pab - \tfrac{1}{2}QAb + \tfrac{1}{2}\Omega(A^2 - a^2), \tag{1}$$

giving

$$\frac{1}{2\pi G\sigma}\frac{dL}{db} = -QA = \frac{1}{G\sigma}\frac{dW}{dA}A,$$

$$\frac{1}{2\pi G\sigma}\frac{dL}{dA} = \Omega A = -\frac{1}{G\sigma}\frac{dW}{db}A,$$

$$\frac{1}{2\pi G\sigma}\frac{dL}{da} = -Pb - \Omega a. \tag{2}$$

and this last is the current function of P, the potential function of the circumference of the circle AB.

On reference to *Trans. A.M.S.*, p. 513, we notice that $\dfrac{L}{2\pi G\sigma p}$ will give the coefficient of mutual induction between the helix, employed in the Ampere Balance of Ayrton and Viriamu Jones, of pitch p, height b, and radius a, and a coaxial circle of radius A in the plane of one end of the helix (*Phil. Trans.*, 1891, 1907).

The current function M of the potential U of the bowl is found to be given by

$$\frac{M}{2\pi G\sigma} = -Pac + QAc + \Omega c^2 \cos\gamma + \Omega' c^2 \cos\phi$$

$$= -\frac{W}{G\sigma}c + \frac{U}{G\sigma}r\cos\phi, \tag{3}$$

satisfying the conditions

$$\left.\begin{array}{l} \dfrac{1}{2\pi G\sigma}\dfrac{dM}{dr} = Qc\sin\phi = A\dfrac{Y_1}{G\sigma} = -A\dfrac{dU}{rd\phi}, \\[3mm] \dfrac{1}{2\pi G\sigma}\dfrac{dM}{rd\phi} = -\Omega'\dfrac{c^2}{r}\sin\phi = -A\dfrac{X_1}{G\sigma} = A\dfrac{dU}{dr}. \end{array}\right\} \tag{4}$$

If N denotes the current function of V, the potential of the solid segment, treated as a homogeneous function of the third degree, it can be determined, in the same manner as V, from

$$3N = r\frac{dN}{dr} + c\frac{dN}{dc} = 2\pi Yr^2\sin\phi + c\frac{dN}{dc}, \tag{5}$$

where

$$\frac{dN}{dc} = M - L\cos\gamma, \tag{6}$$

$$\frac{3N}{2\pi G\rho} = \left(\frac{W}{G\sigma} - S\right)A^2 - \frac{L}{2\pi G\sigma}c\cos\gamma + \frac{Mc}{2\pi G\sigma}; \tag{7}$$

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and this value of N is found to verify in giving

$$\frac{1}{2\pi}\frac{dN}{db} = A\frac{dV}{dA}, \quad \frac{1}{2\pi}\frac{dN}{dA} = -A\frac{dV}{db}. \tag{8}$$

We can now write

$$\frac{V}{G\rho} = -\tfrac{1}{3}\frac{L}{2\pi G\sigma} -\tfrac{1}{3}\frac{W}{G\sigma}(b + c\cos\gamma) + \tfrac{1}{3}\frac{Uc}{G\sigma}$$

$$= \tfrac{1}{3}\frac{U - W\cos\gamma}{G\sigma}c - \tfrac{1}{3}\frac{L}{2\pi G\sigma} - \tfrac{1}{3}\frac{Wb}{G\sigma}. \tag{9}$$

13. In these differentiations it is useful to note that

$$a\frac{dP}{dA} - A\frac{dQ}{dA} - b\frac{d\Omega}{dA} = 0, \quad a\frac{dP}{db} - A\frac{dQ}{db} - b\frac{d\Omega}{db} = 0, \\ a\frac{dP}{da} - A\frac{dQ}{da} - b\frac{d\Omega}{da} = 0, \tag{1}$$

making

$$\frac{1}{G\sigma}\frac{dW}{dA} = -Q, \quad \frac{1}{G\sigma}\frac{dW}{db} = -\Omega, \quad \text{in } \frac{W}{G\sigma} = Pa - QA - \Omega b; \tag{2}$$

and other useful differentiations are:

$$\frac{dP}{dA} = 2\int_0^\pi \frac{-a(A + a\cos\theta)\,d\theta}{PQ^3}, \quad \frac{dP}{db} = 2\int \frac{-ab\,d\theta}{PQ^3}, \\ \frac{dP}{da} = 2\int \frac{(Aa\cos\theta + A^2 + b^2)\,d\theta}{PQ^3}, \tag{3}$$

$$\frac{dQ}{dA} = 2\int \frac{a\cos\theta\,(A + a\cos\theta)\,d\theta}{PQ^3}, \quad \frac{dQ}{db} = 2\int \frac{ab\cos\theta\,d\theta}{PQ^3}, \\ \frac{dQ}{da} = 2\int \frac{-(Aa\cos\theta + A^2 + b^2)\cos\theta\,d\theta}{PQ^3}, \tag{4}$$

$$\frac{d\Omega}{dA} = 2\int \frac{ab\cos\theta\,d\theta}{PQ^3} = \frac{dQ}{db}, \\ \frac{d\Omega}{db} = 2\int \frac{-Aa\cos\theta - a^2}{PQ^3}\,d\theta = -\frac{dQ}{dA} - \frac{Q}{A}, \tag{5}$$

$$a\frac{d\Omega}{da} = -A\frac{d\Omega}{dA} - b\frac{d\Omega}{db} = 2\int \frac{a^2b\,d\theta}{PQ^3}, \quad \frac{d\Omega}{da} = -\frac{dP}{db}. \tag{6}$$

$$a\frac{dP}{da} = -A\frac{dP}{dA} - b\frac{dP}{db}, \quad a\frac{dQ}{da} = -A\frac{dQ}{dA} - b\frac{dQ}{db}. \tag{7}$$

Denoting

$$PA^2 \cdot PB^2 = (A^2 + a^2 + b^2)^2 - 4A^2a^2 \text{ by } D, \tag{8}$$

we find that, expressed by P and Q,

$$\frac{d\Omega}{dA} = \frac{dQ}{db} = -Pa\frac{2Ab}{D} + Qb\frac{A^2 + a^2 + b^2}{D}, \tag{9}$$

$$\frac{d\Omega}{db} = -\frac{dQ}{dA} - \frac{Q}{A} = Pa\frac{A^2 - a^2 - b^2}{D} - QA\frac{A^2 - a^2 + b^2}{D}, \tag{10}$$

$$\frac{dP}{dA} = -PA\frac{A^2 - a^2 + b^2}{D} + Qa\frac{A^2 - a^2 - b^2}{D}, \tag{11}$$

$$\frac{dP}{db} = -Pb\frac{A^2 + a^2 + b^2}{D} + QA\frac{2ab}{D}. \tag{12}$$

Also

$$SA^2 = \tfrac{2}{3}PaA^2 - \tfrac{1}{3}QA(A^2 + a^2 + b^2), \quad S = -\tfrac{1}{3}\frac{D}{Ab}\frac{d\Omega}{dA}, \tag{13}$$

derivable by integration of

$$\frac{d}{d\theta}(Aa\sin\theta \cdot PQ) = -3\frac{A^2a^2\sin\theta}{PQ} + 2\frac{A^2a^2}{PQ} + (A^2 + a^2 + b^2)\frac{Aa\cos\theta}{PQ}, \tag{14}$$

$$\frac{dSA^2}{db} = -QAb, \quad \frac{dSA^2}{dA} = (Pa - QA)A = \frac{WA}{G\sigma} + \Omega Ab. \tag{15}$$

Again, since

$$\frac{1}{G\sigma}\frac{dU}{dr} = \frac{d\Omega}{dr}c + \frac{d\Omega'}{dr}\cdot\frac{c^2}{r} - \Omega'\frac{c^2}{r^2} = -\frac{X_1}{G\sigma} = -\Omega'\frac{c^2}{r^2}, \tag{16}$$

$$\frac{1}{G\sigma}\frac{dU}{rd\phi} = \frac{d\Omega}{rd\phi}c + \frac{d\Omega'}{rd\phi}\frac{c^2}{r} = -\frac{Y_1}{G\sigma} = -Q\frac{c}{r}, \tag{17}$$

it follows that

$$r\frac{d\Omega}{dr} + c\frac{d\Omega'}{dr} = 0, \qquad r\frac{d\Omega}{d\phi} + c\frac{d\Omega'}{d\phi} = -Qr, \tag{18}$$

$$r\frac{d\Omega}{dA} + c\frac{d\Omega'}{dA} = -Q\cos\phi, \qquad r\frac{d\Omega}{db} + c\frac{d\Omega'}{db} = -Q\sin\phi. \tag{19}$$

This collection of formulas is useful for reference in the differentiation of the potential and current function.

14. These expressions for V and N give the potential and current function of a flat lens, and addition or subtraction will give them for a lens, concavo-convex in Fig. 5, or double convex as in Fig. 6.

Then if C_1 is the centre and c_1 the radius of the second spherical surface,

$$\frac{V - V_1}{G\rho} = \tfrac{1}{3} \frac{W}{G\sigma} \, CC_1 + \tfrac{1}{3} \frac{Uc - U_1 c_1}{G\sigma}, \tag{1}$$

for the concave lens of Fig. 5; and in Fig. 6 for the convex lens

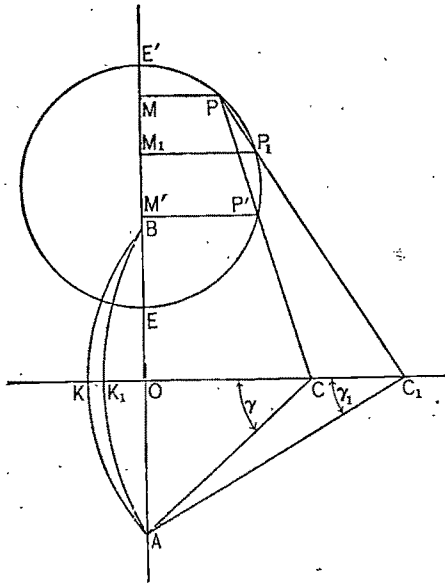$$\frac{V + V_1}{G\rho} = \tfrac{1}{3} \frac{W}{G\sigma} \, CC_1 + \tfrac{1}{3} \frac{Uc + U_1 c_1}{G\sigma}. \tag{2}$$
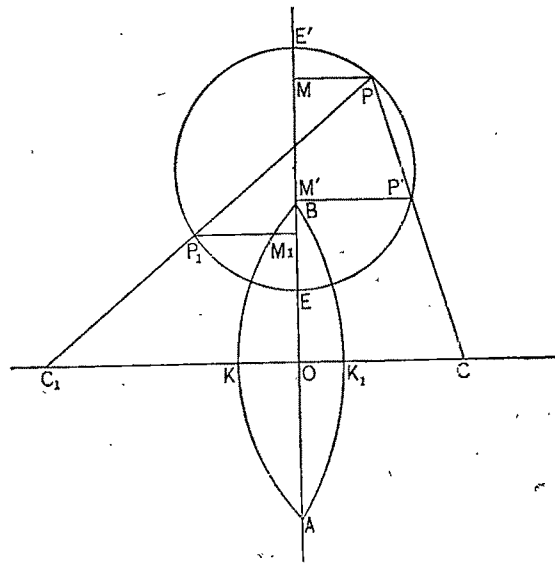

Fig. 5.   Fig. 6.

For example, with a complete sphere, where $CC_1 = 0,\ c = c_1 = a$,

$$\frac{V + V_1}{G\rho} = \tfrac{1}{3} \frac{U + U_1}{G\sigma} \, a, \tag{3}$$

and the cyclic constants of Ω and Ω' must be so adjusted as to make, in the exterior space

$$\frac{U + U_1}{G\sigma} = 4\pi \frac{a^2}{r}, \quad \frac{V + V_1}{G\rho} = \tfrac{4}{3}\pi \frac{a^3}{r}. \tag{4}$$

But inside the sphere

$$\frac{U + U_1}{G\sigma} = 4\pi a, \quad \frac{V + V_1}{G\rho} = 4\pi \left(\tfrac{1}{2} a^2 - \tfrac{1}{6} r^2\right), \tag{5}$$

requiring a careful adjustment of the cyclic constant; herein is the difficulty of expressing the single-valued potential by means of the multiple-valued Ω and Ω'.

Similarly for the current function of the concave lens of Fig. 5,

$$\frac{N - N_1}{2\pi\,G\rho} = \tfrac{1}{3}\frac{L}{2\pi\,G\sigma}\,C C_1 + \tfrac{1}{3}\frac{Mc - M_1 c_1}{2\pi\,G\sigma}, \tag{6}$$

as is evident from (7), § 12.

15. For a thin concave lens,

$$\frac{1}{G\rho}\frac{dV}{d\gamma} = \tfrac{1}{3}\frac{W}{G\sigma}\frac{a}{\sin^2\gamma} + \tfrac{1}{3}\frac{1}{G\sigma}\frac{dUc}{d\gamma}, \tag{1}$$

$$\frac{1}{2\pi\,G\rho}\frac{dN}{d\gamma} = \tfrac{1}{3}\frac{L}{2\pi\,G\sigma}\frac{a}{\sin^2\gamma} + \tfrac{1}{3}\frac{1}{2\pi\,G\sigma}\frac{dMc}{d\gamma}. \tag{2}$$

Keeping A, b, Ω constant, and varying c, r, ϕ, b', A', Ω' with γ, we find

$$\left.\begin{aligned}
\frac{dc}{d\gamma} &= \frac{-a\cos\gamma}{\sin^2\gamma}, & \frac{dr}{d\gamma} &= \frac{-a\cos\phi}{\sin^2\gamma}, & \frac{r\,d\phi}{d\gamma} &= \frac{a\sin\phi}{\sin^2\gamma}, \\
\frac{db'}{d\gamma} &= \frac{A'^2 - a^2 - b'^2}{a}, & \frac{dA'}{d\gamma} &= -\frac{2A'b'}{a},
\end{aligned}\right\} \tag{3}$$

$$\begin{aligned}
\frac{d\Omega'}{d\gamma} &= \frac{d\Omega'}{db'}\frac{db'}{d\gamma} + \frac{d\Omega'}{dA'}\frac{dA'}{d\gamma} \\[4pt]
&= \left(P'a\frac{A'^2 - a^2 - b'^2}{D'} - Q'A'\frac{A'^2 - a^2 + b'^2}{D'}\right)\frac{A'^2 - a^2 - b'^2}{a} \\[4pt]
&\quad + \left(-P'a\frac{2A'b'}{D'} - Q'b'\frac{A'^2 + a^2 + b'^2}{D'}\right)\frac{-2A'b'}{a} \\[4pt]
&= \frac{P'a - Q'A'}{a}, \tag{4}
\end{aligned}$$

$$\frac{1}{G\sigma}\frac{dU}{d\gamma} = -\frac{U}{G\sigma}\frac{\cos\gamma}{\sin\gamma} + \frac{W'}{G\sigma}\frac{a}{r\sin^2\gamma}, \tag{5}$$

$$\frac{1}{G\rho}\frac{dV}{d\gamma} = \tfrac{1}{3}\frac{W}{G\sigma}\frac{a}{\sin^2\gamma} - \tfrac{1}{3}\frac{U}{G\sigma}\frac{2a\cos\gamma}{\sin^2\gamma} + \tfrac{1}{3}\frac{W'}{G\sigma}\frac{ac}{r\sin^2\gamma}. \tag{6}$$

If m denotes the mass of the flat lens,

$$m = \tfrac{1}{3}\rho\pi a^3\frac{(1 - \cos\gamma)^2(2 + \cos\gamma)}{\sin^3\gamma}, \qquad \frac{dm}{d\gamma} = \frac{\rho\pi a^3}{(1 + \cos\gamma)^2}, \tag{7}$$

$$\frac{1}{G\rho}\frac{dV}{d\gamma} = \frac{1}{G}\frac{dV}{dm}\frac{\pi a^3}{(1 + \cos\gamma)^2}. \tag{8}$$

Similarly, from (3), § 12,

$$\frac{1}{2\pi\,G\sigma}\frac{dM}{d\gamma} = -\frac{M\cos\gamma + M'\cos\phi}{2\pi\,G\sigma\sin\gamma} - \frac{U}{G\sigma}\frac{a\sin^2\phi}{\sin^2\gamma}$$

$$= -\frac{M}{2\pi\,G\sigma}\frac{\cos\gamma}{\sin\gamma} + \frac{W'\cos\phi - U}{G\sigma}\frac{a}{\sin^2\gamma}, \qquad (9)$$

$$\frac{1}{2\pi\,G\rho}\frac{dN}{d\gamma} = \tfrac{1}{3}\frac{L}{2\pi\,G\sigma}\frac{a}{\sin^2\gamma} - \tfrac{1}{3}\frac{M}{2\pi\,G\sigma}\frac{2a\cos\gamma}{\sin^2\gamma} + \tfrac{1}{3}\frac{W'\cos\phi - U}{G\sigma}\frac{ac}{\sin^2\gamma}; \qquad (10)$$

and comparing this (10) with (6), we notice that $W'\cos\phi - U$ should be the current function of $\dfrac{W'}{r}$, which proves to be the case, by a differentiation in verification.

16. When $\gamma = 0$, $c, r = \infty$, $\dfrac{c}{r} = 1$, $W = W' = U$, $L = M$,

and $\dfrac{dV}{d\gamma}, \dfrac{dN}{d\gamma}$ take an indeterminate form.

To evaluate them for the lens, which is now a flat disc, in which the thickness and superficial density at a point Q varies as $Q'A \cdot Q'B = a^2 - y^2$ at a radius $OQ' = y$, and for mass μ the superficial density $\sigma = \dfrac{2\mu}{\pi a^4}(a^2 - y^2)$, we calculate the potential v directly by the dissection into ring elements; and then

$$\frac{v}{G\mu} = \frac{1}{G}\frac{dV}{dm} = \frac{2}{\pi a^4}\int_{\theta=0}^{2\pi}\int_{y=0}^{a}(a^2 - y^2)\frac{y\,dy\,d\theta}{PQ'}, \qquad (1)$$

and, with $\gamma = 0$,

$$\frac{1}{G\rho}\frac{dV}{d\gamma} = \frac{1}{G}\frac{dV}{dm}\cdot\frac{\pi a^3}{4}. \qquad (2)$$

Integrating with respect to y, with

$$PQ'^2 = y^2 + 2Ay\cos\theta + A^2 + b^2, \qquad (3)$$

$$\int_0^a(-y^3 + a^2 y)\frac{dy}{PQ'} = -\tfrac{1}{3}PQ^3 + \tfrac{1}{3}PO^3 + \tfrac{3}{2}Aa\cos\theta\cdot PQ$$
$$+ (-\tfrac{5}{2}A^2\cos^2\theta + A^2 + a^2 + b^2)(PQ - PO)$$
$$+ A\cos\theta(A^2 - a^2 - \tfrac{3}{2}b^2 - \tfrac{5}{2}A^2\sin^2\theta)I, \qquad (4)$$

where

$$I = \int_0^a\frac{dy}{PQ'} = \mathrm{ch}^{-1}\frac{PQ}{PZ} - \mathrm{ch}^{-1}\frac{PO}{PZ} = \mathrm{sh}^{-1}\frac{a + A\cos\theta}{PZ} - \mathrm{sh}^{-1}\frac{A\cos\theta}{PZ}, \qquad (5)$$

and PZ is the perpendicular on QO,

$$PZ^2 = A^2 \sin^2 \theta + b^2, \qquad PO^2 = A^2 + b^2, \tag{6}$$

$$\frac{dI}{d\theta} = - \frac{Aa \cos \theta + A^2 + b^2}{PZ^2} \cdot \frac{A \sin \theta}{PQ} + \frac{PO \cdot A \sin \theta}{PZ^2}. \tag{7}$$

Writing

$$A \cos \theta \left(A^2 - a^2 - \tfrac{3}{2} b^2 - \tfrac{5}{2} A^2 \sin^2 \theta \right) I$$

$$= \frac{d}{d\theta} \left[A \sin \theta \left(A^2 - a^2 - \tfrac{3}{2} b^2 - \tfrac{5}{6} A^2 \sin^2 \theta \right) I \right]$$

$$- A \sin \theta \left(A^2 - a^2 - \tfrac{3}{2} b^2 - \tfrac{5}{6} A^2 \sin^2 \theta \right) \frac{dI}{d\theta}, \tag{8}$$

and integrating again with respect to θ, we find after considerable reduction of a previous character that, with $\gamma = 0$,

$$\frac{2a}{G\rho} \frac{dV}{d\gamma} = \tfrac{2}{3} \frac{L}{2\pi G\sigma} b - \tfrac{2}{3} \frac{W}{G\sigma} (A^2 - a^2 - b^2) + \tfrac{2}{3} S A^2; \tag{9}$$

and W can also be calculated in this manner directly, from a dissection of the circle AB into concentric rings.

Thence we infer

$$\frac{2a}{2\pi G\rho} \frac{dN}{d\gamma} = - \frac{L}{2\pi G\sigma} \cdot \tfrac{1}{2} (A^2 - a^2) - \frac{W}{G\sigma} A^2 b + \tfrac{1}{2} S A^2 b. \tag{10}$$

For differentiating, with

$$\frac{1}{2\pi G\sigma} \frac{dL}{db} = \frac{A}{G\sigma} \frac{dW}{dA} = - QA, \qquad \frac{1}{2\pi G\sigma} \frac{dL}{dA} = - \frac{A}{G\sigma} \frac{dV}{db} = \Omega A, \tag{11}$$

we find

$$\frac{2a}{2\pi G\rho} \frac{d^2 N}{db\, d\gamma} = - 2 \frac{W}{G\sigma} A^2 + \tfrac{2}{3} S A^2 = \frac{2aA}{G\rho} \frac{d^2 V}{dA\, d\gamma}, \tag{12}$$

$$\frac{2a}{2\pi G\rho} \frac{d^2 N}{dA\, d\gamma} = - \frac{2LA}{2\pi G\sigma} - 2 \frac{WAb}{G\sigma} = - \frac{2aA}{G\rho} \frac{d^2 V}{db\, d\gamma}. \tag{13}$$

17. Going back again to the concave lens of finite thickness, we find

$$\frac{1}{2\pi G\rho} \frac{d(N - N_1)}{db} = \tfrac{1}{3} A \frac{dW}{dA} \cdot CC_1 + \tfrac{1}{3} A \left(c \frac{dU}{dA} - c_1 \frac{dU_1}{dA} \right)$$

$$= \tfrac{1}{3} QA (b' - b_1) - \tfrac{1}{3} \frac{\Omega' c^3 \sin^3 \phi - \Omega_1 c_1^3 \sin^3 \phi_1}{A}$$
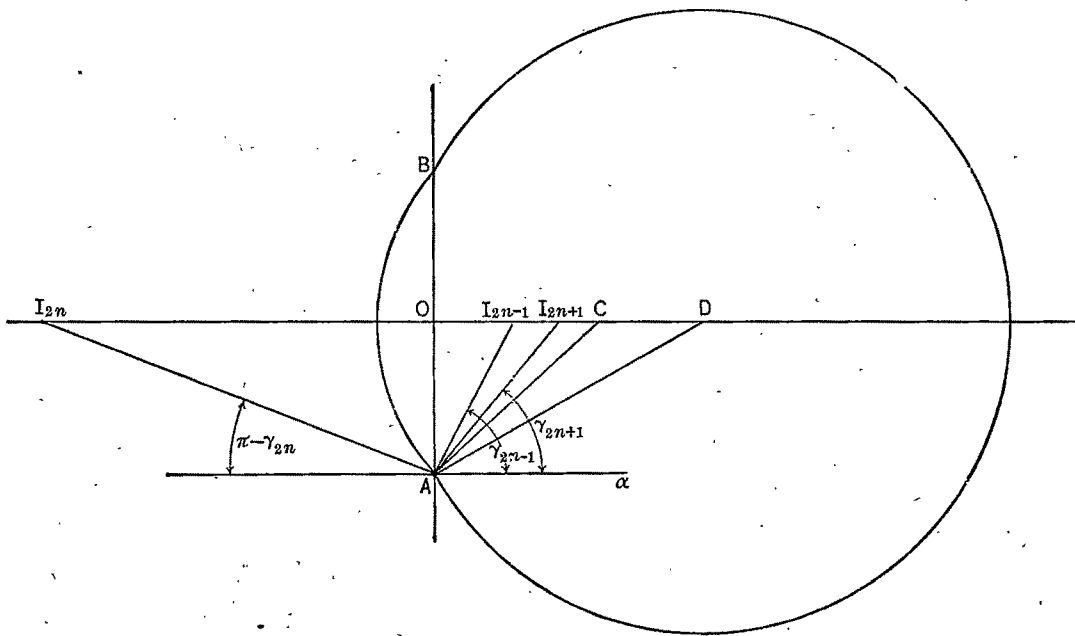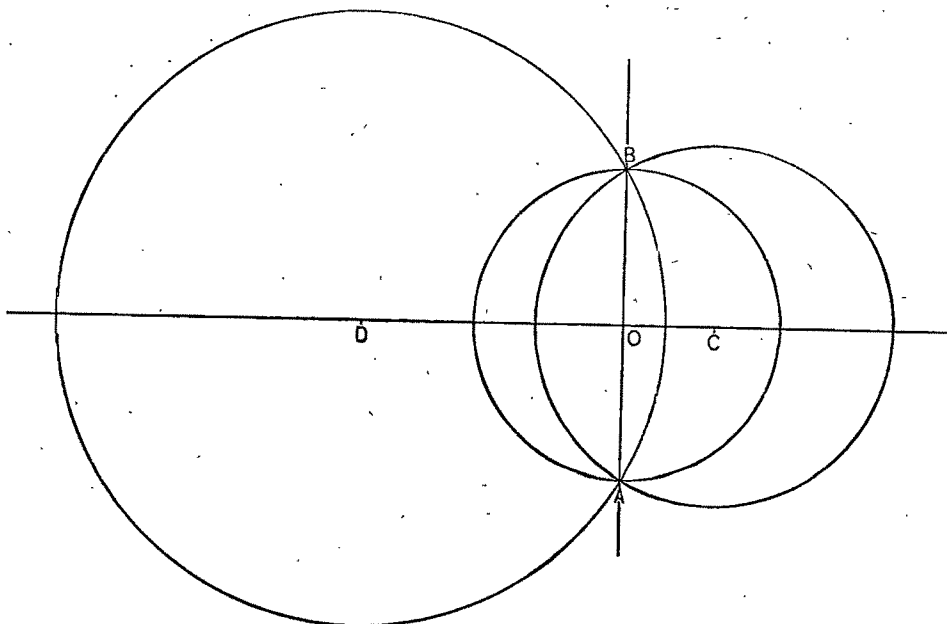
$$= A \frac{d(V - V_1)}{dA}, \tag{1}$$

Fig. 7.

Fig. 8.

$$\frac{1}{2\pi\, G\rho}\frac{d(N - N_1)}{dA} = - \tfrac{1}{3}A\frac{dW}{db}\cdot CC_1 - \tfrac{1}{3}A\left(c\frac{dU}{db} - c_1\frac{dU_1}{db}\right)$$

$$= \tfrac{1}{3}QA(A' - A_1) - \tfrac{1}{3}A\left(\Omega r + \Omega'\frac{c^3}{r^2}\right)\cos\phi$$

$$+ \tfrac{1}{3}A\left(\Omega r_1 + \Omega_1\frac{c_1^3}{r_1^2}\right)\cos\phi_1$$

$$= - A\frac{d(V - V_1)}{db}, \tag{2}$$

giving the components of attraction of the concave lens.

The difficulty is now to return to the flat lens, where $c_1 = \infty$, introducing the indeterminate form $0 \times \infty$; thus the expression for the potential and current function of the flat lens in (7) and (9), §12, requires

$$\mathrm{lt}\,(U_1 - W\cos\gamma_1)\,c_1 = \frac{L}{2\pi} + Wb, \quad \mathrm{lt}\,\frac{L\cos\gamma_1 - M_1}{2\pi}\,c_1 = \left(\frac{W}{G\sigma} - S\right)A^2; \tag{3}$$

which are not obvious, and require careful treatment.

18. From the expression in (6), §15, it should not be impossible now to obtain the potential of the thin curved lens from an integration of its ring-shaped elements, and so obtain a direct method which employs the dissection into elements symmetrical about the axis OC.

The expressions in (1), (2), §17, will give the magnetic potential of the lens, and its current function of vector potential, when magnetized uniformly; and by analogy with the ellipsoid in §11, the same expressions should help towards the determination of the velocity function when the lens moves through liquid, like the bob of a pendulum cutting the air; but some modification is required still, before the requisite conditions are satisfied.

Take two spheres in Fig. 7, centre C and D, intersecting in the circle AB at an angle $\gamma - \delta = 2\alpha$, γ and δ denoting the angle OCA and ODA; and put $\gamma + \delta = 2\beta$, the bisector of the angle DAC making an angle β with OC.

Next take the succession of points I_1, I_2, ..., I_n, ... on CD which are the electric images or inverse points in succession with respect to the two spherical surfaces, denoted by the centre C and D; beginning with I_1 at C, and I_2 the image or inverse point of I_1 in the sphere D, I_3 the image of I_2 in the sphere C, and so on; and denote the angle $O I_n A$ or $I_n N\alpha$ by γ_n.

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Then since

$$DI_{2n-1} \cdot DI_{2n} = DA^2, \quad \text{and} \quad CI_{2n} \cdot CI_{2n+1} = CA^2, \tag{1}$$

$$OI_{2n}A = CAI_{2n-1} = DAI_{2n+1}, \tag{2}$$

$$\pi - \gamma_{2n} = \gamma_{2n-1} - \delta = \gamma_{2n+1} - \gamma, \tag{3}$$

$$\left.\begin{aligned} \gamma_{2n+1} - \gamma_{2n-1} &= \gamma - \delta = 2\alpha, \\ \gamma_{2n+1} - \gamma_1 &= 2n\alpha, \end{aligned}\right\} \tag{4}$$

and with $\gamma_1 = \gamma = \alpha + \beta$,

$$\left.\begin{aligned} \gamma_{2n+1} &= (2n+1)\alpha + \beta, \\ \gamma_{2n} &= \pi - 2n\alpha. \end{aligned}\right\} \tag{5}$$

If f_n denotes the distance of I_n to the right of O in Fig. 7, and C_n denotes the distance CI_n, c_1 being c, the radius of the sphere C,

$$f_{2n} = a\cot(\pi - 2n\alpha) = -a\cot 2n\alpha, \quad f_{2n+1} = a\cot[(2n+1)\alpha + \beta], \tag{6}$$

$$c_{2n} = a\csc 2n\alpha, \quad c_{2n+1} = a\csc[(2n+1)\alpha + \beta]. \tag{7}$$

If $J_1, J_2, \ldots, J_n, \ldots$ denotes the series of images, beginning with J_1 at D, then by a change of sign of α, caused by the interchange of γ and δ,

$$g_{2n} = a\cot 2n\alpha, \quad g_{2n+1} = -a\cot[(2n+1)\alpha - \beta], \tag{8}$$

$$d_{2n} = a\csc 2n\alpha, \quad d_{2n+1} = a\csc[(2n+1)\alpha - \beta], \tag{9}$$

with f and c changed into g and d.

If the spheres cut at an angle $2\alpha = \dfrac{\pi}{m}$, or $\dfrac{r\pi}{m}$, an aliquot part of π, the number of images is finite and $2m + 1$, analogous to the finite number of images seen by reflexion in two plane mirrors at an angle $\pi r/m$.

Thus if m is an even number $2p$, $f_p = g_p = 0$, $f_{p+r} = g_{p-r}$; and if m is odd, $2p + 1$,

$$f_{2p+1} = g_{2p-1}, \quad f_{2p+r} = g_{2p-r}.$$

(R. A. Herman, "A Problem in Fluid Motion," *Quarterly Journal of Mathematics*, Vol. XXII, 1887.)

Denoting the distance PI_n, PJ_n by r_n, s_n, consider first the potential function

$$U = \frac{c_1}{r_1} - \frac{c_2}{r_2} + \frac{c_3}{r_3} \cdots + \frac{d_3}{s_3} - \frac{d_2}{s_2} + \frac{d_1}{s_1}, \tag{10}$$

due to the series of a finite number of images I_n and J_n, beginning with I_1 at C and J_1 at D, and joining up in the middle.

Over the sphere C,

$$r_1 = c_1, \quad \frac{c_2}{r_2} = \frac{c_3}{r_3}, \quad \ldots, \quad \frac{d_2}{r_2} = \frac{d_1}{r_1}; \tag{11}$$

and over the sphere D,

$$s_1 = d_1, \quad \frac{d_2}{s_2} = \frac{d_3}{s_3}, \quad \ldots, \quad \frac{c_2}{r_2} = \frac{c_1}{r_1}; \tag{12}$$

so that over the surface of the sphere, $U = 1$, and U will serve for the potential in exterior space of an electrical distribution over the outside surface of the intersecting spheres.

Change the circular functions into hyperbolic, to obtain the electrification of two non-intersecting spheres, with images along the line of centres AB, A and B being the limiting points of the two spheres, and then Maxwell's coefficients of induction and capacity (*E. and M.*, § 173) are given by

$$q_{aa} = \sum_{n=1}^{\infty} c_{2n-1}, \quad q_{ab} = -\Sigma c_{2n} = -\Sigma d_{2n}, \quad q_{bb} = \Sigma d_{2n-1}; \tag{13}$$

the images being now infinite in number, but condensed ultimately at A and B.

Next consider the potential function V and its current function N, given by

$$\frac{V}{G\rho} = \frac{c_1^3}{r_1} - \frac{c_2^3}{r_2} - \ldots - \frac{d_2^3}{s_2} + \frac{d_1^3}{s_1} = \Sigma(-1)^{n-1}\left(\frac{c_n^3}{r_n} + \frac{d_n^3}{s_n}\right), \tag{14}$$

$$\frac{N}{2\pi G\rho} = c_1^3 \cos PI_1O - c_2^3 \cos PI_2O - \ldots - d_2^3 \cos PJ_2O + d_1^3 \cos PJ_1O$$

$$= \Sigma(-1)^{n-1}\left(c_n^3 \frac{f_n - x}{r_n} + d_n^3 \frac{g_n - x}{s_n}\right), \tag{15}$$

due to a series of spheres, of density $\pm \frac{3}{4}\rho$, centres at I_n and J_n, all intersecting in the same circle AB, like a series of lenses; a sphere being condensed afterwards into a particle at its centre.

Then with these particles replaced by magnetic molecules,

$$\frac{1}{2\pi G\rho}\frac{dN}{dx} = -y^2 \Sigma(-1)^{n-1}\left(\frac{c_n^3}{r_n^3} + \frac{d_n^3}{s_n^3}\right), \tag{16}$$

reducing, as in (11) and (12), to $-y^2$ over the sphere C and D; so that for a velocity u of the two spheres in the direction OC through infinite liquid, the current function ψ and velocity function ϕ will be given by

$$\frac{\psi}{\frac{1}{2}u} = \frac{1}{2\pi G\rho}\frac{dN}{dx}, \quad \frac{\phi}{\frac{1}{2}u} = \frac{1}{G\rho}\frac{dV}{dx}. \tag{17}$$

The kinetic energy T of the exterior liquid, taken of density σ, will be found by integrating over the spherical surfaces; and will be given by

$$\frac{T}{\frac{1}{2}u^2} \text{ (the effective inertia)} = \tfrac{1}{2}\sigma\int \frac{\phi}{\frac{1}{2}u}\cos PCO\, dS$$

$$= \tfrac{1}{2}\sigma\int \frac{1}{G\rho}\frac{dV}{dx}\, 2\pi y\, dy = -\tfrac{1}{2}\sigma\int \frac{1}{G\rho}\frac{dN}{dy}\, dy$$

$$= -\tfrac{1}{2}\sigma\left[\frac{N}{G\rho}\right] + \int \frac{\sigma}{G\rho}\frac{dN}{dx}\, dx = -\tfrac{1}{2}\sigma\left[\frac{N}{G\rho}\right] - \sigma\int \pi y^2\, dx; \quad (18)$$

and in passing from the left at H to the right at K of the spheres, N increases by $2\,G\rho\pi\,(c_1^3 - c_2^3 - \ldots - d_2^3 + d_1^3)$; so that

$$\text{the effective inertia} = \sigma\pi\,(c_1^3 - c_2^3 - \ldots - d_2^3 + d_1^3)$$
$$- \text{ the displacement of liquid.} \quad (19)$$

Thus when the spheres coalesce, $m = 1$, $c_1 = d_1 = c$, and

$$\text{the effective inertia} = 2\sigma\pi c^3 - \tfrac{4}{3}\sigma\pi c^3 = \tfrac{2}{3}\sigma\pi c^3$$
$$= \text{ half the displacement of liquid;} \quad (20)$$

a verification.

For two orthogonal spheres, as in Fig. 8, $m = 2$, and there is only one image I_2 or J_2 at O, $c_2 = d_2 = a$; and V is the potential of three lenses, an interior double-convex lens, and two exterior concavo-convex, with two interspaces. This is analogous to the case of the squarable meniscus of Antiphon and Hippocrates (Simplicius, F. Rudio, Teubner, 1907).

19. In the reduction in (16), § 2, of

$$\Omega = \int_0^{2\pi} \frac{QN^2}{PN^2}\cdot\frac{b\, d\theta}{PQ} \quad (1)$$

to Legendre's standard form, and its numerical expression by his Table IX, the consideration of imaginary parameters is avoided by the substitution

$$\left.\begin{array}{l} \dfrac{PN^2}{QN^2} = \dfrac{s-\sigma}{\sigma-s_3}, \qquad \dfrac{PQ^2}{QN^2} = \dfrac{s-s_3}{\sigma-s_3}, \\[2ex] \dfrac{QN^2}{PQ^2} = \dfrac{\sigma-s_3}{s-s_3} = \dfrac{a^2\sin^2\theta}{2Aa\cos\theta + A^2 + a^2 + b^2}. \end{array}\right\} \quad (2)$$

This is equivalent to a quadric transformation of Mr. Hill's reduction; and the quadratic for $\cos\theta$ is

$$\left.\begin{aligned}
&\cos^2\theta + 2\frac{A}{a}\frac{\sigma-s_3}{s-s_3}\cos\theta + \frac{A^2+a^2+b^2}{a^2}\frac{\sigma-s_3}{s-s_3} - 1 = 0,\\
&\left(\cos\theta + \frac{A}{a}\frac{\sigma-s}{s-s_3}\right)^2 = 1 - \frac{A^2+a^2+b^2}{a^2}\frac{\sigma-s_3}{s-s_3} + \frac{A^2}{a^2}\left(\frac{\sigma-s_3}{s-s_3}\right)^2\\
&\qquad\qquad = \frac{s-s_1\cdot s-s_2}{(s-s_3)^2},
\end{aligned}\right\} \quad (3)$$

with

$$\left.\begin{aligned}
\frac{r_1^2+r_2^2}{2a^2} &= \frac{A^2+a^2+b^2}{a^2} = \frac{s_1-s_3}{\sigma-s_3} + \frac{s_2-s_3}{\sigma-s_3},\\
\frac{r_1^2-r_2^2}{2a^2} &= 2\frac{A}{a} = 2\frac{\sqrt{(s_1-s_3\cdot s_2-s_3)}}{\sigma-s_3},\\
\frac{r_1,\,r_2}{a} &= \frac{\sqrt{(s_1-s_3)} \pm \sqrt{(s_2-s_3)}}{\sqrt{(\sigma-s_3)}},\\
\frac{r_1+r_2}{2a} &= \sqrt{\frac{s_1-s_3}{\sigma-s_3}}, \quad \frac{r_1-r_2}{2a} = \sqrt{\frac{s_2-s_3}{\sigma-s_3}},\\
\frac{b}{a} &= \frac{\sqrt{(s_1-\sigma\cdot\sigma-s_2)}}{\sigma-s_3}.
\end{aligned}\right\} \quad (4)$$

$$\left.\begin{aligned}
\cos\theta &= \frac{-\sqrt{(s_1-s_3\cdot s_2-s_3)} + \sqrt{(s-s_1\cdot s-s_2)}}{s-s_3},\\
\sin\theta &= \frac{\sqrt{(s_2-s_3\cdot s-s_1)} + \sqrt{(s_1-s_3\cdot s-s_2)}}{s-s_3},\\
\frac{PQ}{a} &= \sin\theta\sqrt{\frac{s-s_3}{\sigma-s_3}} = \frac{\sqrt{(s_2-s_3\cdot s-s_1)} + \sqrt{(s_1-s_3\cdot s-s_2)}}{\sqrt{(s-s_3\cdot\sigma-s_3)}},\\
-\sin\theta\frac{d\theta}{ds} &= \frac{[\sqrt{(s_2-s_3\cdot s-s_1)} + \sqrt{(s_1-s_3\cdot s-s_2)}]^2}{2(s-s_3)^2\sqrt{(s-s_1\cdot s-s_2)}},\\
-\frac{d\theta}{ds} &= \frac{\sqrt{(s_2-s_3\cdot s-s_1)} + \sqrt{(s_1-s_3\cdot s-s_2)}}{2(s-s_3)\sqrt{(s-s_1\cdot s-s_2)}},\\
\frac{a\,d\theta}{PQ} &= -\frac{\sqrt{(\sigma-s_3)}\,ds}{\sqrt{S}}.
\end{aligned}\right\} \quad (5)$$

As θ increases from 0 to π, s diminishes from ∞ to s_1 and increases again to ∞, so that, as in *Trans. A. M. S.*, p. 491,

$$\Omega = 4\int_{s_1}^{\infty} \frac{\sqrt{(s_1-\sigma\cdot\sigma-s_2\cdot\sigma-s_3)}}{s-\sigma}\frac{ds}{\sqrt{S}} = 2\pi f - 4K\mathrm{zn}(1-f)K', \quad (6)$$

the equivalent of Legendre's formula (m'), *Fonctions elliptiques*, p. 138, with $\frac{1}{4}\Omega$ the equivalent of the left-hand side of Legendre's equation, when we put Legendre's

$$\sin^2\phi = \frac{s_1 - s_3}{s - s_3}, \quad \text{and his } n = -\Delta^2(\theta, k'), \quad n\sin^2\phi = -\frac{\sigma - s_3}{s - s_3}, \tag{7}$$

$$\left. \begin{aligned} \sin^2\theta &= \operatorname{sn}^2(1-f)K' = \frac{s_1 - \sigma}{s_1 - s_2}, \quad \operatorname{cn}^2(1-f)K' = \frac{\sigma - s_2}{s_1 - s_2}, \\ \operatorname{dn}^2(1-f)K' &= \frac{\sigma - s_3}{s_1 - s_3} = \frac{4a^2}{(r_1+r_2)^2}, \quad x = \sqrt{\frac{s_2 - s_3}{s_1 - s_3}} = \frac{r_1 - r_2}{r_1 + r_2}; \end{aligned} \right\} \tag{8}$$

and then

$$\operatorname{dn}fK' = \frac{r_1 - r_2}{2a}, \quad \operatorname{sn}^2fK' = \frac{1 - \operatorname{dn}^2fK'}{x'^2} = \frac{(r_1+r_2)^2 - 4A^2}{4r_1r_2} = \sin^2\chi, \tag{9}$$

denoting the angle PEE' by χ, so that $\chi = \operatorname{am}fK'$, and then if PB cuts the circle EE' again in p, $\operatorname{am}(1-f)K' = E'Ep$. For

$$\frac{AE}{BE} = \frac{AE'}{BE'} = \frac{r_1}{r_2}, \quad \frac{OE'}{OA} = \frac{r_1 + r_2}{r_1 - r_2} = \frac{1}{x}, \quad \frac{OE}{OA} = \frac{r_1 - r_2}{r_1 + r_2} = x, \tag{10}$$

$$\frac{EE'}{a} = \frac{1}{x} - x = \frac{4r_1r_2}{r_1^2 - r_2^2} = \frac{r_1r_2}{Aa}, \tag{11}$$

$$\left. \begin{aligned} E'M &= OE' - OM = a\frac{r_1 + r_2}{r_1 - r_2} - A = \frac{(r_1+r_2)^2}{4A} - A, \\ \sin^2\chi &= \frac{E'M}{EE'} = \frac{(r_1+r_2)^2 - 4A^2}{4r_1r_2}. \end{aligned} \right\} \tag{12}$$

As P moves round the circle $E'PE$, x and K do not change, but f increases from 0 to 1, and Ω from 0 at E' to 2π at E.

Expressed in terms of $E'\chi$ and $F'\chi$ of Legendre's Table IX, the accent denoting the comodulus x',

$$fK' = F'\chi, \quad \operatorname{zn}fK' = E'\chi - fH', \tag{13}$$

$$\operatorname{zn}(1-f)K' = x'^2\frac{\sin\chi\cos\chi}{\Delta'\chi} - \operatorname{zn}fK', \tag{14}$$

so that, employing Legendre's relation,

$$\tfrac{1}{2}\pi = KH' + K'H - KK', \tag{15}$$

$$\tfrac{1}{4}\Omega = H \cdot F'\chi - K(F'\chi - E'\chi) - Kx'^2\frac{\sin\chi\cos\chi}{\Delta'\chi}; \tag{16}$$

or, more simply, denoting the angle $E'Ep$ by θ,

$$\tfrac{1}{4}\Omega = \tfrac{1}{2}\pi + (K - H)F'\theta - KE'\theta, \tag{17}$$

as in Legendre's equation (m'), p. 138.

Denoting the angle $P'EE'$ by $\chi' = \operatorname{am} f' K'$,

$$\Omega' = 2\pi f' - 4 K \operatorname{zn}(1 - f') K'; \tag{18}$$

and Ω, Ω' have a cyclic constant 2π for a circuit of the circle $E'PE$.

But as P and P' describe the circle in opposite direction, the cyclic constants cancel in the expression of V; otherwise, if a cyclic constant could exist in a gravity potential, work could be obtained from a circuit, or in the popular expression, perpetual motion would be possible.

The potential and attraction is a single-valued function, but expressed by Ω and Ω', functions having a cyclic constant 4π for a circuit of the circle EE'; and care must be taken to adjust the cyclic constants at any point P, although they cancel when P has made a circuit of the circle EE' in one direction, and P' in the opposite.

Thus as P moves in the direction PE, and Ω increases from zero at E', Ω may be taken to have gained 4π in passing through E', and 4π must be deducted, so that Ω should be zero again at E' at the end of the circuit.

So, too, P' moves in the opposite direction, starting from zero at E''; and Ω' must lose 4π, and have 4π added in passing through E.

Compare the gain or loss of a day in going round the world, moving east or west.

20. According to this reduction in §19 of the elliptic integral

$$P = \int_0^\pi \frac{2a\,d\theta}{PQ} = 4\sqrt{\frac{\sigma - s_3}{s_1 - s_3}} \int_{s_1}^\infty \frac{\sqrt{(s_1 - s_3)}\,ds}{\sqrt{S}}$$

$$= 4 K \operatorname{dn}(1 - f) K' = \frac{8 K a}{r_1 + r_2}; \tag{1}$$

and with

$$s - s_3 = (s_1 - s_3)\frac{1}{\operatorname{sn}^2 u}, \qquad s - s_2 = (s_1 - s_3)\frac{\operatorname{dn}^2 u}{\operatorname{sn}^2 u},$$

$$s - s_1 = (s_1 - s_3)\frac{\operatorname{cn}^2 u}{\operatorname{dn}^2 u}, \qquad \frac{\sqrt{(s_1 - s_3)}\,ds}{\sqrt{S}} = du, \tag{2}$$

$$\cos\theta = -x\operatorname{sn}^2 u + \operatorname{cn} u\operatorname{dn} u, \qquad \sin\theta = (x\operatorname{cn} u + \operatorname{dn} u)\operatorname{sn} u; \tag{3}$$

and u grows from 0 to $2K$ as θ increases from 0 to π, so that

$$Q = \int_0^\pi \frac{-2a\cos\theta\,d\theta}{PQ} = 2\sqrt{\frac{\sigma - s_3}{s_1 - s_3}} \int_0^{2K} (x\operatorname{sn}^2 u - \operatorname{cn} u\operatorname{dn} u)\,du$$

$$= 4\frac{\operatorname{dn}(1 - f) K'}{x} \int_0^K (1 - \operatorname{dn}^2 u)\,du$$

$$= \frac{4}{\operatorname{dn} f K'}(K - H) = \frac{8a}{r_1 - r_2}(K - H) = 2\frac{r_1 + r_2}{A}(K - H), \tag{4}$$

H denoting the complete elliptic integral of the second kind, to modulus x; and then

$$\int_0^\pi \frac{2\,Q\,N^2}{PQ}\,d\theta = 2\,a \int \sin^2\theta\,\frac{a\,d\theta}{PQ}$$

$$= 2\,a \,\operatorname{dn}(1-f)\,K' \int_0^{2K} (x\,\operatorname{cn} u + \operatorname{dn} u)^2 \operatorname{sn}^2 u\,d u$$

$$= \frac{8\,a^2}{r_1 + r_2} \int_0^K (x^2\,\operatorname{cn}^2 u + \operatorname{dn}^2 u)\,\operatorname{sn}^2 u\,d u$$

$$= \frac{8\,a^2}{r_1 + r_2} \cdot \frac{(1 + x^2)\,H - (1 - x^2)\,K}{3\,x^2}. \tag{5}$$

21. In Maxwell's notation, *E. and M.*, § 701,

$$M = 2\,\pi\,Q\,A = 4\,\pi\,(K - H)\,(r_1 + r_2), \qquad x = \frac{r_1 - r_2}{r_1 + r_2}, \tag{1}$$

his second expression when corrected; but if a return is made to the modulus c employed in Maxwell's first transformation, and in Mr. Hill's reduction, a preparation is made of the expression of Ω by means of the theorems in § 47, (1), (4), (9), p. 505, *Trans. A. M. S.*, 1907, where with

$$I_4 = \sin^{-1} \frac{Q\,N\cdot P\,M}{P\,N\cdot M\,Q} = \cos^{-1} \frac{M\,N\cdot P\,Q}{P\,N\cdot M\,Q}$$

$=$ complement of the angle between the planes PQN, PQM, (2)

a differentiation gives

$$\frac{d\,I_4}{d\theta} = \frac{Q\,N^2}{P\,M^2}\cdot\frac{b}{PQ} + \frac{A\,a\cos\theta + a^2}{M\,Q^2}\cdot\frac{b}{PQ}$$

$$= \tfrac{1}{2}\frac{d\,\Omega}{d\theta} + \frac{\tfrac{1}{2}\,b}{PQ} - \frac{\tfrac{1}{2}(A^2 - a^2)}{M\,Q^2}\cdot\frac{b}{PQ}, \tag{3}$$

which exhibits the addition of the imaginary parameters in Mr. Hill's treatment.

The reduction to a standard form is made, without any rearrangement of the constituents, by the substitution (*Trans. A. M. S.*, § 26, p. 477):

$$\left.\begin{aligned}
PQ^2 = r^2 &= A^2 + 2\,A\,a\cos\theta + a^2 + b^2 = m^2(t_1 - t),\\
PA^2 = r_1^2 &= (A + a)^2 + b^2 = m^2(t_1 - t_3),\\
PB^2 = r_2^2 &= (A - a)^2 + b^2 = m^2(t_1 - t_2),\\
r_1^2 - r^2 &= 2\,A\,a\,(1 - \cos\theta) = m^2(t - t_3),\\
r^2 - r_2^2 &= 2\,A\,a\,(1 + \cos\theta) = m^2(t_2 - t),\\
2\,A\,a\sin\theta &= m^2\sqrt{(t_2 - t\cdot t - t_3)}, \quad 2\,A\,a\sin\theta\,d\theta = m^2\,dt,\\
d\theta &= \frac{dt}{\sqrt{(t_2 - t\cdot t - t_3)}},\quad \frac{d\theta}{PQ} = \frac{dt}{m\sqrt{(t_1 - t\cdot t_2 - t\cdot t - t_3)}} = \frac{2\,dt}{m\sqrt{T}}.
\end{aligned}\right\} \tag{4}$$

r now denoting PQ, and m a homogeneity factor; and then

$$P = \int_0^\pi \frac{2\,a\,d\theta}{PQ} = \frac{4\,a}{m} \int_{t_3}^{t_2} \frac{dt}{\sqrt{T}} = \frac{4\,a}{r_1} \int \frac{\sqrt{(t_1 - t_3)}\,dt}{\sqrt{T}} = \frac{4\,Fa}{r_1}, \qquad (5)$$

where F denotes the elliptic quarter period to the modulus c (not to be confused with the radius of the spherical surface), where

$$c^2 = \frac{t_2 - t_3}{t_1 - t_3} = \frac{4\,A\,a}{r_1^2}, \qquad c' = \frac{r_2}{r_1}; \qquad (6)$$

and

$$M = 2\,\pi\,QA = 2\,\pi \int_0^\pi \frac{-2\,A\,a\cos\theta\,d\theta}{PQ} = 2\,\pi\,m \int_{t_3}^{t_2} (2t - t_2 - t_3)\,\frac{dt}{\sqrt{T}}$$

$$= 2\,\pi\,m \int [t_1 - t_3 + t_1 - t_2 - 2\,(t_1 - t)]\,\frac{dt}{\sqrt{T}}. \qquad (7)$$

We now put

$$t_1 - t = (t_1 - t_3)\,\mathrm{dn}^2 u, \quad t_2 - t = (t_2 - t_3)\,\mathrm{cn}^2 u, \quad t - t_3 = (t_2 - t_3)\,\mathrm{sn}^2 u, \quad (8)$$

so that

$$\theta = 2\,\omega = 2\,\mathrm{am}\,(u,\,c), \quad \omega = ABQ, \text{ on Fig. 3}, \qquad (9)$$

$$M = 2\,\pi\,r_1 \int_0^K (1 + c'^2 - 2\,\mathrm{dn}^2 u)\,du$$

$$= 2\,\pi\,r_1\,[(2 - c^2)\,F - 2\,E], \qquad (10)$$

agreeing with Maxwell's first expression for M in *E. and M.*, § 701, when his sign is changed; E denotes here the complete second elliptic integral to the modulus c.

Other useful formulas are:

$$\int_0^{2\pi} PQ\,d\theta = 4\,r_1 \int_0^{\frac{1}{2}\pi} \Delta\omega\,d\omega = 4\,E\,r_1, \qquad (11)$$

$$P = \frac{4\,Fa}{r_1} = \frac{8\,Ka}{r_1 + r_2}, \quad F = (1 + x)\,K, \quad K = \tfrac{1}{2}\,(1 + c')\,F, \qquad (12)$$

$$P\,\frac{b}{a} = 4\,K x'^2\,\frac{\mathrm{sn}\,fK'\,\mathrm{cn}\,fK'}{\mathrm{dn}\,fK'} = 4\,Fc'\,\mathrm{sn}\,2\,fF'. \qquad (13)$$

22. When P is close to the circle AB, at B, r_2 is small and P is large; writing it

$$P = \int_0^{2\pi} \frac{a\, d\theta}{PQ} = \int_0^{\frac{1}{2}\pi} \frac{4a\, d\omega}{PQ} = \int \frac{4a \sin\omega\, d\omega}{PQ} + \int \frac{4a(1 - \sin\omega)\, d\omega}{PQ}, \quad (1)$$

the first integral

$$\int_\omega^{\frac{1}{2}\pi} \frac{4a \sin\omega\, d\omega}{PQ} = \frac{4a}{\sqrt{(r_1^2 - r_2^2)}} \operatorname{ch}^{-1} \frac{PQ}{r_2}, \quad \int_0^{\frac{1}{2}\pi} \frac{4a \sin\omega\, d\omega}{PQ} = 2\sqrt{\frac{a}{A}} \operatorname{ch}^{-1} \frac{r_1}{r_2}$$

$$= 2\sqrt{\frac{a}{A}} \log \frac{r_1 + 2\sqrt{(Aa)}}{r_2}, \quad \text{ultimately } 2 \log \frac{4a}{r_2}, \quad (2)$$

with $r_1 = 2a$, $A = a$; and

$$\int_0^{\frac{1}{2}\pi} \frac{4a(1 - \sin\omega)\, d\omega}{PQ} < \frac{4a}{r_1} \int \frac{1 - \sin\omega}{\cos\omega}\, d\omega = 2 \int \frac{\cos\omega}{1 + \sin\omega}\, d\omega \text{ or } 2 \log 2, \quad (3)$$

so that, when r_2 is small, we may take

$$P = 2 \log \frac{8a}{r_2} + \text{small terms}, \quad (4)$$

$$P - Q = \int_0^{2\pi} \frac{a(1 + \cos\theta)\, d\theta}{PQ}$$

$$= \frac{8a}{r_1} \int_0^{\frac{1}{2}\pi} \frac{\cos^2\omega\, d\omega}{\Delta\omega} < \frac{8a}{r_1} \int \cos\omega\, d\omega \text{ or } \frac{8a}{r_1}, \quad (5)$$

and replacing r_1 by $2a$,

$$Q = 2 \log \frac{8a}{r_2} - 4 + \text{small terms} = 2 \log \frac{8a}{e^2 r_2} + \text{small terms}, \quad (6)$$

$$M = 2\pi QA = 4\pi a \log \frac{8a}{e^2 r_2} + \text{small terms}. \quad (7)$$

23. In this reduction of the third elliptic integral, we put

$$\left. \begin{array}{l} MQ^2 = A^2 + 2Aa \cos\theta + a^2 = m^2(\tau - t), \quad PM^2 = b^2 = m^2(t_1 - \tau), \\ MA^2 = (A + a)^2 = m^2(\tau - t_2), \quad MB^2 = (A - a)^2 = m^2(\tau - t_3), \end{array} \right\} \quad (1)$$

and with

$$\operatorname{sn}^2 2fF' = \frac{t_1 - \tau}{t_1 - t_2} = \frac{b^2}{r_1^2} = \sin^2\psi, \quad \psi = \operatorname{am}(2fF', c'), \quad (2)$$

where ψ denotes the angle PBE', growing from 0 to π as f increases from 0 to 1,

$$\int_0^\pi \frac{\frac{1}{2}(A^2 - a^2)}{MQ^2} \cdot \frac{b\, d\theta}{PQ} = \int_{t_3}^{t_2} \frac{\sqrt{(t_1 - \tau \cdot \tau - t_2 \cdot \tau - t_3)}}{\tau - t} \cdot \frac{dt}{\sqrt{T}}$$

$$= \pi f + F \operatorname{zn} 2fF', \quad (3)$$

in accordance with Legendre's equation (m'); thus from (3), § 21,

$$\Omega = 2I_4 + \int_{t_3}^{t_2} \frac{2\sqrt{(t_1 - \tau \cdot \tau - t_2 \cdot \tau - t_3)}}{\tau - t} \frac{dt}{\sqrt{T}} - \int \frac{2\sqrt{(t-\tau)}\,dt}{\sqrt{T}}$$

$$= 2I_4 + 2\pi f + 2\,F\,\mathrm{zn}\,2f\,F' - 2\,Fc'\,\mathrm{sn}\,2f\,F', \tag{4}$$

agreeing with (6), § 19, provided $I_4 = 0$, and

$$2\,Fc'\,\mathrm{sn}\,2f\,F' - 2\,F\,\mathrm{zn}\,2f\,F' = 4\,K\,\mathrm{zn}\,(1-f)\,K', \tag{5}$$

$$2\,Fc'\,\mathrm{sn}\,2f\,F' + 2\,F\,\mathrm{zn}\,2f\,F' = 4\,K\,\mathrm{zn}\,f\,K', \tag{6}$$

a theorem of the quadric transformation of the zeta function.

The result of (2), § 12, will represent the current function or lines of magnetic force of the electric current circulating round the circumference AB, or of the circular plate AB magnetized normally; and this magnetic plate is equivalent to a compound plate composed of the superposition of two thin plates of superficial density $\pm\,\sigma$.

When these two plates are drawn apart on the same axis to a distance b, the result is equivalent to an integration with respect to b, and L is obtained for the pair of end-plates, or for the cylinder magnetized longitudinally; also for the equivalent solenoid made by a current sheet of electricity circulating circumferentially, giving the same magnetic field as the helical current of the Ampère balance.

In the hydrodynamical interpretation, the difference of the two values of L for an end of the solenoid will give the lines of flow of liquid, circulating through the solenoidal tube.

Then, with the former method of integration by parts,

$$\frac{L}{2\pi G\sigma} = \int_0^b -QA\,db = \int_0^b \int_0^{2\pi} \frac{Aa\cos\theta\,d\theta\,db}{PQ} = \int_0^{2\pi} Aa\,\cos\theta\,\mathrm{sh}^{-1}\frac{b}{MQ}\,d\theta$$

$$= \left[Aa\sin\theta\,\mathrm{sh}^{-1}\frac{b}{MQ}\right]_0^{2\pi} - \int \frac{A^2 a^2 \sin^2\theta}{MQ^2} \cdot \frac{b\,d\theta}{PQ}$$

$$= \int \left[\tfrac{1}{2}Aa\cos\theta - \tfrac{1}{4}(A^2 + a^2) + \tfrac{1}{4}\frac{(A^2 - a^2)^2}{MQ^2}\right]\frac{b\,d\theta}{PQ}, \tag{7}$$

which agrees with (1), § 12, when the result of (3), § 23, is employed.

24. Denoting the integral in (3), § 23, by $II(MQ)$, then by addition of (3), § 4, and (2), § 21,

$$J_4 = J_3 + I_4 = - II(MQ) + \int \frac{\frac{1}{2} b\, d\theta}{PQ} + \int \frac{c\,(r-x)(c\cos\phi - x\cos\gamma)}{c^2 - x^2} \cdot \frac{d\theta}{PQ}, \quad (1)$$

$$J_4 = \sin^{-1}\frac{PQ \cdot NQ}{LQ \cdot MQ}\sin\phi = \cos^{-1}\frac{(2A\cos\phi + b\sin\phi)a\cos\theta + (A^2 + a^2)\cos\phi + Ab\sin\phi}{LQ \cdot MQ}$$

$=$ the complement of the angle between the planes PQM, PQL. (2)

Similarly for the point P', and

$$II(M'Q) = \int \frac{\frac{1}{2}(A'^2 - a^2)}{M'Q^2} \cdot \frac{b'\, d\theta}{PQ} = \pi f' + F\,zn\,2f'\,F', \quad (3)$$

$$J_4' = J_3' + I_4' = - II(M'Q) + \int \frac{\frac{1}{2} b'\, d\theta}{PQ}$$

$$+ \int \frac{(c^2 - rx)(c\cos\phi - x\cos\gamma)}{c^2 - x^2} \cdot \frac{d\theta}{PQ}. \quad (4)$$

Thence, by addition and subtraction,

$$J_4 + J_4' + II(MQ) + II(M'Q)$$

$$= \tfrac{1}{2}(b + b')\int \frac{d\theta}{PQ} + \int \frac{(r + c)(c\cos\phi - x\cos\gamma)}{c + x} \cdot \frac{d\theta}{PQ}$$

$$= [\tfrac{1}{2}(b + b') - (r + c)\cos\gamma]\int \frac{d\theta}{PQ}$$

$$+ \int \frac{c\,(r + c)(\cos\phi + \cos\gamma)}{c + x} \cdot \frac{d\theta}{PQ}, \quad (5)$$

$$J_4 - J_4' + \Omega(MQ) - \Omega(M'Q)$$

$$= \tfrac{1}{2}(b - b')\int \frac{d\theta}{PQ} + \int \frac{(r - c)(c\cos\phi - x\cos\gamma)}{c - x} \cdot \frac{d\theta}{PQ}$$

$$= [\tfrac{1}{2}(b - b') + (r - c)\cos\gamma]\int \frac{d\theta}{PQ}$$

$$+ \int \frac{c\,(r - c)(\cos\phi - \cos\gamma)}{c - x} \cdot \frac{d\theta}{PQ}; \quad (6)$$

thus showing the addition and subtraction of the parameters of the third elliptic integrals, $\Omega(c)$ and $\Omega(-c)$, where

$$\Omega(c) = \int \frac{c\,(r - c)(\cos\gamma - \cos\phi)}{c - x} \frac{d\theta}{PQ},$$

$$\Omega(-c) = \int \frac{c\,(r + c)(\cos\gamma + \cos\phi)}{c + x} \frac{d\theta}{PQ}. \quad \left.\right\} \quad (7)$$

Expressed by the variable $r = PQ$, or by $\Delta\omega$, with

$$R = r_1^2 - r^2 \cdot r^2 - r_2^2, \qquad r = r_1 \Delta\omega, \qquad \omega = \tfrac{1}{2}\theta, \tag{8}$$

and reinstating Mr. Hill's x',

$$\Omega\,(c) = \int \frac{2\,c\,x'\,(x' - c)\,(\cos\gamma - \cos\phi)}{r^2 - (x' - c)^2} \cdot \frac{dr}{\sqrt{R}}$$

$$= \int \frac{2\,c\,x'\,(x' - c)\,(\cos\gamma - \cos\phi)}{r_1^2 \Delta^2 \omega - (x' - c)^2} \cdot \frac{1}{r_1} \frac{d\omega}{\Delta\omega}, \tag{9}$$

$$\Omega\,(-c) = \int \frac{2\,c\,x'\,(x' + c)\,(\cos\gamma + \cos\phi)}{(x' + c)^2 - r^2} \cdot \frac{dr}{\sqrt{R}}$$

$$= \int \frac{2\,c\,x'\,(x' + c)\,(\cos\gamma + \cos\phi)}{(x' + c)^2 - r_1^2 \Delta^2 \omega} \cdot \frac{1}{r_1} \frac{d\omega}{\Delta\omega}, \tag{10}$$

two elliptic integrals of the third kind, in Legendre's normal form, to a modulus c as in §§ 21, 23, not to be confused with the radius of the spherical surface.

The exploration of the field can be carried out for simple aliquot values of f, such as $f = \tfrac{1}{2}, \tfrac{1}{3}, \tfrac{2}{3}, \tfrac{1}{4}, \tfrac{3}{4}, \tfrac{1}{6}, \ldots$, when $\operatorname{zn} fK'$ can be expressed as an algebraical function, as shown in *Trans. A. M. S.*, § 60, p. 521.

Denoting the angle PAE' by ψ',

$$\sin\psi' = c' \sin\psi, \quad \cos\psi' = \operatorname{dn} 2fF', \quad \frac{\tan\psi'}{\tan\psi} = \frac{A - a}{A + a} = c' \operatorname{sn} (1 - 2f) F'; \tag{11}$$

$$\Omega = 2\pi f + 2 F \operatorname{zn} 2fF' - 2 F \sin\psi'. \tag{12}$$

The stereographic coordinates u and v are useful to employ, such that

$$b + A i = a \tan\tfrac{1}{2}(u + vi), \qquad b, A = a \frac{\sin u, \operatorname{sh} v}{\operatorname{ch} v + \cos u}, \tag{13}$$

$$e^v = \frac{AP}{PB} = c', \qquad u = APB = \psi - \psi' = \pi - 2\theta, \qquad \psi + \psi' = 2\chi, \tag{14}$$

since PE bisects APB; and $a \csc u$ is the radius of the circle round ABP.

Thence the formulas of the quadric transformation,

$$\psi = \tfrac{1}{2}\pi - \theta + \chi, \qquad \sin\psi = \cos(\theta - \chi) = (1 + x) \frac{\sin\chi \cos\chi}{\Delta\chi}, \tag{15}$$

$$\operatorname{sn} 2fF' = (1 + x) \frac{\operatorname{sn} fK' \operatorname{cn} fK'}{\operatorname{dn} fK'}. \tag{16}$$

Thus when $f = \frac{1}{2}$, $A = a$, $\psi = \frac{1}{2}\pi$, $\sin\psi' = c'$, so that APB is the modular angle for c; and from (4), (5), (6), § 23, and (12), § 21,

$$4\,K\,\mathrm{zn}\,\tfrac{1}{2}\,K' = 2\,F\,c', \qquad \mathrm{zn}\,\tfrac{1}{2}\,K = \tfrac{1}{2}\,(1 - x),$$

$$\Omega = \pi - 2\,F\,c' = \pi - 2\,K\,(1 - x),$$

$$\frac{L}{2\pi\,G\,\sigma} = \tfrac{1}{2}\,(P + Q)\,ab = ab \int_0^\pi \frac{a\,(1 - \cos\theta)\,d\theta}{PQ} = \frac{4\,a^2\,b}{r_1}\int_0^{\frac{1}{2}\pi} \frac{\sin^2\omega}{\Delta\omega}\,d\omega$$

$$= \frac{4\,a^2\,b}{r_1}\,\frac{F - E}{c^2} = (F - E)\,b\,r_1 = (F - E)\,c'\,r_1^2;$$

and when $b = 0$, $c' = 0$, $F = \infty$, but $F'c = 0$, $\Omega = \pi$, $L = 0$.

When $f = \frac{1}{3}$ (*Phil. Trans.*, 1904, p. 261),

$$\cos\psi = \frac{p - 1}{p + 1}, \qquad \cos\psi' = \frac{p - 1}{2}, \qquad \mathrm{zn}\,2f\,F' = \frac{-p + 3}{6}\sqrt{p}\;(3 > p > 1),$$

$$\sec\psi - \sec\psi' = 1, \text{ equivalent to } \sin\chi + \cos\theta = 1.$$

When $f = \frac{1}{4}$ (*Phil. Trans.*, p. 276),

$$\cos^2\psi = \frac{c}{1 + c}, \qquad \cos^2\psi' = c, \qquad \mathrm{zn}\,2f\,F' = \tfrac{1}{2}\,(1 - c), \qquad \tan\psi = \sec\psi'.$$

When $f = \frac{1}{6}$ (*Trans. A. M. S.*, p. 522),

$$\sin\psi = \frac{2}{p + 1}, \qquad \sin^2\psi' = \frac{(p + 1)(-p + 3)}{4p}, \qquad \mathrm{zn}\,2f\,F' = \frac{-p^2 + 9}{12\sqrt{p}},$$

$$\tan^2\psi - \tan^2\psi' = \frac{1 - \sin\psi}{1 + \sin\psi}, \qquad \tan^2\psi' = \frac{2\sin\psi - 1}{\cos^2\psi}.$$